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PHYSICAL SOCIETY OF LONDON.

From December 1917 *to August* 1918.

VOL. XXX.

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PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1917–1918.

October 26, 1917.

Meeting held at Imperial College of Science.

Mr. W. R. **Cooper**, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. " On a Class of Multiple Thin Objectives." By T. **Smith**, B.A.

2. " On the Radius of the Electron and the Nuclear Structure of Atoms." By Prof. J. W. **Nicholson**, F.R.S.

November 9, 1917.

Meeting held at Imperial College of Science.

Prof. C. V. **Boys**, F.R.S., President, in the Chair.

The President and Mr. R. S. Whipple referred to the loss the Society had sustained by the death of Mr. W. Duddell, F.R.S.

The following Papers were read :—

1. " On the Thermo-Electric Properties of Fused Metals." By Messrs. C. R. **Darling** and A. W. **Grace**.

2. " Triple Cemented Objectives." By Mr. T. **Smith**, B.A., and Miss A. B. **Dale**.

November 23, 1917.

Meeting held at Imperial College of Science.

Mr. W. R. Cooper, M.A., Vice-President, in the Chair.

The following Paper was read :—

"Some Problems of Stability of Atoms and Molecules." By Prof. J. W. Nicholson, F.R.S.

An Exhibition of the Uses of Certain Methods of Classification in Optics was given by Mr. T. H. Blakesley, M.A.

January 25, 1918.

Meeting held at Imperial College of Science.

The Presidential Address was given by Prof. C. V. Boys.

Annual General Meeting.

February 8, 1918.

Prof. C. V. Boys, F.R.S., President, in the Chair.

The Annual Report of the Council was taken as read.

In the year 1917 eleven ordinary Meetings were held. The average attendance at the Meetings was 29.

The Third Guthrie Lecture was delivered at the Meeting of March 23, 1917, by Prof. Langevin. The subject of the lecture was "Molecular Orientation."

The annual Exhibition of Apparatus was again suspended owing to the War.

The number of Honorary Fellows on the roll on December 31, 1917, was 10, and the number of Ordinary Fellows 430. Nine new Fellows were elected, and there were no resignations.

The Society has to mourn the loss of Mr. W. Duddell, F.R.S., who for many years had taken an active part in the work of the Society. The Society also has to record with regret the deaths of Mr. Walter Baily, Dr. C. V. Burton, Sir W. D. Niven, F.R.S., Mr. Wilson Noble, Mr. S. T. Preston, Prof. G. Quincke, the Rev. P. R. Sleeman, and the Hon Sir James Stirling, F.R.S.

The Council has appointed representatives on the Nitrogen Products Committee under the Ministry of Munitions, on the Board of Scientific Societies, and on a Committee of the Röntgen Society for the discussion of points of interest to the X-ray industry. The Council have also appointed a Committee to consider and report upon the possibility of steps being taken to improve the professional status of the physicist.

The Treasurer's Report was read by Mr. W. R. Cooper.

The accounts for the past year show a less satisfactory position than usual, the expenditure exceeding the revenue by £48. 17s. 9d. The expenditure, however, includes a sum of £15. 15s. 10d. for secretarial expenses in respect of the year 1916, so that the debit balance, strictly speaking, amounts to £33. 1s. 11d.

The expenditure on printing is considerably more than in the previous year, and the donation to the S. P. Thompson memorial is an unusual item.

The unsatisfactory position, however, is really due to the fact that many Fellows are in arrears with their subscriptions. The balance sheet shows that the sum thus due to the Society at December 31 last was £219. 18s. 6d., which is very heavy compared with the corresponding figure of other years. Owing to war conditions and the absence of many Fellows on active service, this was to be expected to some extent; but even so the amount seems abnormal. As it is improbable that all the arrears will be realised I have thought it desirable to set off a reserve of £31. 10s. against the full sum.

The investments, which have been valued at market prices through the courtesy of Parr's Bank, have depreciated slightly below the figure in the last balance sheet.

Both reports were unanimously adopted.

THE TREASURER IN ACCOUNT WITH THE PHYSICAL SOCIETY, FROM JANUARY 1ST, 1917, TO DECEMBER 31ST, 1917.

INCOME AND EXPENDITURE ACCOUNT.

Dr.	£	s.	d.	£	s.	d.
Entrance Fees	10	10	0			
Subscriptions by Fellows	401	2	0			
„ Voluntary	16	16	0			
„ by Students	1	11	6			
„ Arrears paid	18	18	0			
„ Paid in Advance	2	7	6			
„ for "Science Abstracts" and Advance Proofs	10	2	6			
				461	7	6
Composition Fees				Nil.		
Dividends :—						
Furness Debenture Stock	11	19	10			
Midland Railway	30	10	0			
Metropolitan Board of Works	5	5	0			
Lancaster Corporation Stock	9	0	0			
New South Wales	6	13	6			
London, Brighton & South Coast Railway	19	5	2			
Great Eastern Railway	15	0	0			
India 3½% Stock	13	2	8			
Exchequer Bonds, 5%	20	0	0			
				130	16	2
Interest on deposit account				4	0	0
Sales of Publications (Fleetway Press, Ltd.)				158	7	9
Balance, being excess of expenditure over income				48	17	9
				£803	9	2

Cr.	£	s.	d.	£	s.	d.
"Science Abstracts"	274	16	0			
„ „ Extra copies	2	16	6			
				277	12	6
Fleetway Press, Ltd. :—						
"Proceedings"	295	0	7			
Bulletin	13	13	4			
Distribution	46	18	7			
General	28	12	10			
				384	5	4
Periodicals	1	15	0			
Reporting	36	17	6			
Refreshments and Attendance	13	5	7			
Guthrie Lecture—Honorarium	10	0	0			
Royal Asiatic Society	2	2	0			
Petty Cash—						
Secretaries' Expenses (including £15 15s. 10d. for 1916)	28	1	6			
Treasurer's Expenses	1	14	8			
Bank Charges and Cheques		8	11			
Insurance	6	10	0			
				100	15	2
Donations—						
S. P. Thompson Memorial	25	0	0			
International Catalogue	5	16	2			
Conjoint Board of Scientific Societies	10	0	0			
				40	16	2
				£803	9	2

W. R. COOPER, *Acting Honorary Treasurer.*

Audited and found correct,

January 31st, 1918.

T. MATHER,
A. A. C. SWINTON, } *Honorary Auditors.*

BALANCE SHEET AT DECEMBER 31ST, 1917.

ASSETS.	£	s.	d.	£	s.	d.
Subscriptions in arrears	219	18	6			
Less reserve for subscriptions probably unrealisable	31	10	0			
				188	8	6
Investments (valued at Dec. 31):—						
£533 Furness 3 per cent. Debenture Stock	277	0	0			
£1,600 Midland Railway 2½ per cent. Perpetual Preference Stock	734	0	0			
£200 Metropolitan Board of Works 3½ per cent. Consolidated Stock	164	0	0			
£400 Lancaster Corporation 3 per cent. Redeemable Stock	228	0	0			
£254 2s. 9d. New South Wales 3½ per cent. Ordinary Stock	218	0	0			
£500 London, Brighton & South Coast Railway Ordinary Stock	365	0	0			
£500 Great Eastern Railway 4 per cent. Debenture Stock	361	0	0			
£500 India 3½ per cent. Stock	315	0	0			
£400 Exchequer 5 per cent. Bonds, 1921	399	0	0			
				3,061	0	0
Outstanding Credit on Sales				59	3	10
Outstanding Credit (overcharge)				3	16	8
Stock of Publications (Treasurer's estimate)				220	0	0
Cash at Bank on Deposit				100	0	0
Cash at Bank, Current Account at Dec. 31	45	10	2			
Adjustment for outstanding cheques	3	12	0			
				49	2	2
Cash in hand (Treasurer's Petty Cash)					5	4
				£3,681	16	6

LIABILITIES.	£	s.	d.
Life Compositions	1,880	10	0
Sundry Creditors	11	10	0
Balance, General Fund	1,789	16	6
	£3,681	16	6

W. R. COOPER, *Acting Honorary Treasurer.*

Audited and found correct,

T. MATHER,
A. A. C. SWINTON, } *Honorary Auditors*

January 31, 1918.

LIFE COMPOSITION FUND.

	£	s.	d
151 Fellows paid £10	1,510	0	0
3 Fellows paid £15	45	0	0
5 Fellows paid £21	105	0	0
7 Fellows paid £31. 10s.	220	10	0
	£1,880	10	0

Audited and found correct,

W. R. COOPER, *Acting Honorary Treasurer.*

T. MATHER,
A. A. C. SWINTON, } *Honorary Auditors.*

January 31st, 1918.

After the customary votes of thanks, the election of Officers and Council took place, the new Council being constituted as follows :—

President.—Prof. C. H. Lees, D.Sc., F.R.S.

Vice-Presidents, who have filled the office of President.—Prof. G. C. Foster, D.Sc., LL.D., F.R.S. ; Prof. R. B. Clifton, M.A., F.R.S. ; Prof. A. W. Reinold, C.B., M.A., F.R.S. ; Sir W. de W. Abney, R.E., K.C.B., D.C.L., F.R.S. ; Prin. Sir Oliver J. Lodge, D.Sc., LL.D., F.R.S. ; Sir R. T. Glazebrook, C.B., D.Sc., F.R.S. ; Prof. J. Perry, D.Sc., F.R.S. ; C. Chree, Sc.D., LL.D., F.R.S. ; Prof. H. L. Callendar, M.A., LL.D., F.R.S. ; Prof. A. Schuster, Ph.D., Sc.D., F.R.S. ; Sir J. J. Thomson, O.M., D.Sc., F.R.S. ; Prof C. Vernon Boys, F.R.S.

Vice-Presidents.—Prof. J. W. Nicholson, M.A., D.Sc., F R S ; Prof. O. W. Richardson, M.A., D.Sc., F.R.S. ; S. W. J. Smith, M.A., D.Sc., F.R.S. ; W. E. Sumpner, D.Sc.

Secretaries.—Prof. W. Eccles, D.Sc. ; H. S. Allen, M.A., D.Sc.

Foreign Secretary.—Sir R. T. Glazebrook, C.B., D.Sc., F.R.S.

Treasurer.—W. R. Cooper, M.A., B.Sc.

Librarian.—S. W. J. Smith, M.A., D.Sc., F.R.S.

Other Members of Council.—Prof. E. H. Barton, D.Sc., F.R.S. ; C. R. Darling, F.I.C. ; Prof. G. W. O. Howe, D.Sc. ; D. Owen, D.Sc. ; C. C. Paterson ; C. E. S. Phillips, F.R.S.E. ; S. Russ, M.A., D.Sc. ; T. Smith, B.A. ; F. J. W. Whipple, M.A.

The President announced that Dr. R. S. Willows had resigned his secretaryship on leaving London for a post in the north. The Council had appointed Dr. H. S. Allen, M.A., in his place.

At the conclusion of the general business Prof. Boys vacated the chair, which was taken by Prof. Lees. The following Papers were read :—

1. "A Recording Thermometer." By Prof. C. V. Boys, F.R.S.

2. "The Primary Monochromatic Aberrations of a Centred Optical System." By Mr. S. D. Chalmers, M.A.

February 22, 1918.

Meeting held at Imperial College of Science.

Prof. C. H. Lees, F.R.S., President, in the Chair.

The following Papers were read :—

1. "A Note on the Use of Approximate Methods in Obtaining Constructional Data for Telescope Objectives." By T. Smith, B.A.

2. "A Suggestion as to the Origin of Spectral Series." By Dr. H. S. Allen, M.A.

March 8, 1918.

Meeting held at Imperial College of Science.

Prof. C. H. Lees, F.R.S., President, in the Chair.

The following Papers were read :—

1. "The Asymmetrical Distribution of Corpuscular Radiation from X-rays." By E. A. Owen, M.Sc.

2. "On 'Air-Standard' Internal Combustion Engines and their Efficiencies." By Prof. C. H. Lees, F.R.S.

March 22, 1918.

Meeting held at Imperial College of Science.

Prof. C. H. Lees, F.R.S., President, in the Chair.

The Fourth Guthrie Lecture was delivered by Prof. J. C. McLennan, of Toronto, who took as his subject "The Origin of Spectra."

April 26, 1918.

Meeting held at Imperial College of Science.

Prof. C. H. LEES, F.R.S., President, in the Chair.

The following Papers were read :—

1. "Notes on the Pulfrich Refractometer." By J. GUILD, A.R.C.S.

2. "On the Accuracy Attainable with Critical Angle Refractometers." By F. SIMEON, B.Sc.

3. *"Cohesion" (Fourth Paper). By Prof. H. CHATLEY.

* Taken as read in the absence of the Author.

May 10, 1918.

Meeting held at Imperial College of Science.

Prof. C. H. LEES, F.R.S., President, in the Chair.

The following Papers were read :—

1. "The Times of Sudden Commencement of Magnetic Storms." By Dr. S. CHAPMAN.

2. "The Entropy of a Metal." By Dr. H. S. ALLEN.

3. "On Tracing Rays through an Optical System." By T. SMITH, B.A.

June 14, 1918.

Meeting held at Imperial College of Science.

Prof. C. H. LEES, F.R.S., President, in the Chair.

A Discussion took place on "The Teaching of Physics in Schools." It was opened by Sir OLIVER LODGE, F.R.S., and taken part in by Prof. R. A. Gregory, Mr. C. L. Bryant, Dr. T. J. Baker, Mr. C. E. Ashford, Mr. A. T. Simmons, Prof. F. Womack, Mr. J. Nicol, Mr. E. Smith and Mr. F. B. Stead.

June 28, 1918.

Meeting held at Imperial College of Science.

Prof. C. H. **Lees**, F.R.S., President, in the Chair.

The following Paper was read :—

"A New Method of Measuring Alternating Currents and Electric Oscillations." By Mr. I. **Williams**, B.Sc.

A Demonstration of Coupled Vibrations was given by Prof. E. H. **Barton**, F.R.S., and Miss H. M. **Browning**.

I. *The Radius of the Electron, and the Nuclear Structure of Atoms.* *By Prof.* J. W. Nicholson, *M.A., D.Sc., F.R.S.*

Received Aug. 22, 1917.

The present note is intended rather to make a suggestion than to formulate any definite theory of the structure of the nucleus of an atom according to the model at present found necessary in order to interpret such phenomena as radioactivity, atomic number, and scattering of charged particles by atoms. The electron is usually regarded as a kind of globule of electricity with a definite radius, and as the nuclei of the more complex atoms must, from certain considerations, be supposed to contain electrons, and at the same time preserve their minute size, a difficulty is encountered unless we may suppose that electrons and positive charges can actually in some way inter-penetrate each other and occupy the same space. Some means of removing the finite radius of an electron, and with it all discontinuity at a prescribed surface, is manifestly desirable. On theories such as that of Lorentz, the electron, a sphere when at rest, is deformed when in motion, but in all cases in which hypotheses as to the inner structure and internal equilibrium of an electron are introduced, it has been given, when at rest, this definite "radius," marking off a distinct region from the aether, and involving a discontinuity of some form at the boundary. However great the departure of this view from the more natural intuitions or prejudices of those who regard the electron as a structure built out of aether, it has been of great service at many points. Especially, in the hands of Lorentz and others, it has led to a conception of the variation of the mass of an electron with its speed, which is in undoubted agreement with careful experiments, and must contain a large substratum of truth. Such considerations involve the existence of a linear constant which is the same for every electron and is usually regarded as a "radius." A similar constant is, of course, necessary for the elementary positive charge.

If the terminology which makes use of an aether, out of which the elementary charges are constructed as regions of strain, is adopted, it would seem more natural that such line-constants should be constants with some significance throughout the whole aether, rather than constants which only come into being when the aether is strained into the form of matter. The aether may, in fact, be in some manner cellular, with these linear magnitudes involved in the specification of the cells, and thereby in any strained structure composed from them. In this Paper we accordingly make a tentative suggestion towards this form of interpretation of the line-constants by regarding the electron as a state of strain which is for practical purposes concentrated at its "centre," rapidly diminishing outwards from this point according to some very convergent law, which involves a line-constant in its specification. It is found that no important difference is made in the mutual reactions of electrons except at distances comparable with their "radii," and that if the strains are regarded as capable of superposition, inter-penetration is readily possible without the introduction of indefinitely large forces between the components. A form in which neutral doublets could exist as a part of nuclear structure also becomes evident.

The mathematical treatment, on the basis of a simple exponential law of attenuation of the strain, is only an illustration developed for purposes of clearness. If the suggestion were to correspond in any way with reality, no phenomena at present available could give a clue to the actual law. An elementary argument on the basis of physical dimensions, however, is sufficient to show that any other law would only lead to certain differences in numerical coefficients.

Sir Joseph Larmor alone appears not to be definitely bound to the point of view of the finite electron. For the purposes of his theory,[*] an electron is a type of singularity, made up of aether in a peculiar state of strain, which needs no more precise mathematical definition of the state of strain than is implied in the relation

$$\iint (lf + mg + nh)\,ds = e,$$

or, the surface integral of aethereal polarisation over any surface surrounding one electron is $4\pi e$. This serves as a definition of e. In another form, an electron is a region in which the divergence of the aethereal polarisation is not zero. From this point of view, an electron might have no boundary, and the singularity could be confined effectively to a very small region by choosing a distribution of electrical density ρ, defined by

---

[*] "Æther and Matter," Camb. Univ. Press, 7900, pp. 86-90.

$$\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}=\rho,$$

following any rapidly convergent law of variation, for example, $\rho=\varepsilon^{-\lambda r}$, where r is distance from a point. The departure of the surface integral from $4\pi e$ would be inappreciable within a few diameters of the electron, if a diameter is defined as a length comparable with λ^{-1}.

It is noteworthy that Lorentz, whose electron, of those investigated in detail, alone seems to give a reasonable description of those physical phenomena for which the nature of the electron is important, should have found it necessary to impose a contraction on the moving electron of like magnitude with the contraction of all the " interspaces " between electrons, found earlier to the second order by Larmor, by an analysis which is equally applicable to all orders. This equality of the contraction, whether regarded from the point of view of the Principle of Relativity or not, seems in itself to imply a continuity of the electron with the rest of space—an aethereal structure for the electron with no defined boundary. A subsequent brief survey of the consequences of the exponential formula already suggested, as a mere illustrative case, will indicate the bearing of this remark.

But a length is necessarily associated with an electron, as is even apparent from the existence of its electromagnetic mass, with its necessary dimensions. Such a length in the exponential case would be given by λ^{-1}. Whatever the physical significance of this length, it is evident that a view which attaches to it the same significance for all parts of an infinite aether is more acceptable on some grounds than one which relates it to a definite region only round the " centre " of the electron. The endowment of the whole aether with such a linear constant is ultimately equivalent to an endowment with some form of structure. Such a possibility of aethereal structure might ultimately solve many difficulties. It appears to be demanded by the recent quantum or unit theory of energy—provided that the view of an aether is retained—and it would involve an equality of λ for all electrons, since λ would no longer be a property of the electron, but of the whole aether. There is evidence that e, the charge of an electron, bears a relation to Planck's quantum constant h. These brief indications will suffice to show that we can find a means of removing the boundary of an electron while retaining all the analysis of Lorentz, so that a reconciliation of the quantum theory with the necessary properties of the electron, and, in fact, a deduction of the one from the other, is a possible hope for the future. The analysis of the bounded electron, which becomes spheroidal when in motion, then appears as a convenient mathematical substitute for the more complete analysis of an electron of infinite extent, consisting of a state of strain impressed on a structural aether, but a strain of such a mathematical form that it is confined for physical purposes in a certain region of minute extent, comparable in diameter with a linear magnitude involved in the structure of the aethereal cells.

When, in the ordinary view, the radius of an electron is a, the usual formula for the mass of the electron, of electromagnetic origin, is

$$m_0=\frac{2}{3}\frac{e^2}{ac^2}$$

for speeds small in comparison with c, the velocity of light in free aether. In the present note, we shall not be concerned with a higher approximation on account of the comparatively small velocities which it is necessary to ascribe to the electrons in the model atoms which are found capable of giving some account of such phenomena as spectral series.

We proceed to a brief survey of the illustrative case of a more continuous electron, already indicated in the preceding pages. It is not claimed that this is more than a mere illustration of the possibilities of such a theory.

Let us define an electron-centre as an origin around which there is a symmetrical distribution of polarisation satisfying

$$\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}=\rho=A\epsilon^{-\lambda r},$$

where λ is a constant, the reciprocal of a certain length fixed by the aethereal structure. In polar co-ordinates—

$$\frac{1}{r^2}\frac{\partial}{\partial r}(fr^2)+\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}(g\sin\theta)+\frac{1}{r}\sin\theta\frac{\partial h}{\partial\phi}=A\epsilon^{-\lambda r},$$

where (fgh) are now radial and transversal components. In case of a dependence only on r,

$$\frac{\partial}{\partial r}(fr^2)=Ar^2\epsilon^{-\lambda r},$$

and we may write $g=h=o$.

Thus $$f r^2 = -A\left\{\frac{r^2}{\lambda} + \frac{2r}{\lambda^2} + \frac{2}{\lambda^3}\right\}\varepsilon^{-\lambda r} + \text{constant}.$$

With the constant equal to $2A/\lambda^3$, f is finite at the origin, and in fact zero, so that

$$f = -A\left\{\frac{1}{\lambda} + \frac{2}{r\lambda^2} + \frac{2}{r^2\lambda^3}\right\}\varepsilon^{-\lambda r} + \frac{2A}{\lambda^3 r^2},$$

and at a great distance, the intensity of electric force follows the law of inverse square.

The total charge of the electron being e,

$$e = \iiint \rho\, dx dy dz$$

taken throughout space, or

$$e = A\int_0^\infty\int_0^{2\pi}\int_0^\pi \varepsilon^{-\lambda r} r^2 \sin\theta dr d\varphi d\theta$$

$$= 4\pi A\int_0^\infty r^2\varepsilon^{-\lambda r} dr,$$

$$= 8\pi A/\lambda^3,$$

so that at a sufficient distance

$$f = \frac{e}{4\pi r^2},$$

and the electric force is e/r^2. The true polarisation would be

$$f = \frac{e}{4\pi r^2} - \frac{e}{8\pi}\left\{\frac{2}{r^2} + \frac{2\lambda}{r} + \lambda^2\right\}\varepsilon^{-\lambda r}.$$

The electrostatic energy in the field becomes

$$W = \frac{1}{8\pi}\iiint (4\pi f)^2 r^2 d\omega dr,$$

or, after considerable reduction, at various stages

$$W = \frac{1}{2}e^2\int_0^\infty dr\left\{\frac{1}{r} - \left(\frac{1}{r} + \lambda + \frac{\lambda^2 r}{2}\right)\varepsilon^{-\lambda r}\right\}^2,$$

$$= \frac{\lambda e^2}{32} - \frac{\lambda e^2}{8} + \frac{1}{2}e^2\int_0^\infty \frac{dr}{r^2}\left\{1 - (1+\lambda r)\varepsilon^{-\lambda r}\right\}^2,$$

$$= -\frac{3}{32}\lambda e^2 + \lambda^2 e^2\int_0^\infty dr\varepsilon^{-\lambda r}\left\{1 - (1+\lambda r)\varepsilon^{-\lambda r}\right\}$$

(integrating by parts)

$$= -\frac{3}{32}\lambda e^2 + \frac{\lambda e^2}{4} = \frac{5}{32}\lambda e^2.$$

A more difficult problem is the determination of the mutual energy of two electrons occupying the aether at the same time, so that their fields are superposed. From the solution, a knowledge of a change in the law of force between the electrons at distances comparable with λ, can be obtained.

If for convenience we write

$$rf(r)=\frac{1}{r}-\left(\frac{1}{r}+\lambda+\frac{\lambda^2 r}{2}\right)\varepsilon^{-\lambda r},$$

and (r, r') are distances measured from the electrons, which are in the z axis, the portion of the field energy which is mutual is—

$$W_1=\frac{e^2}{4\pi}\iiint f(r)f(r')d\tau,$$

where $d\tau$ is an element of volume, or

$$W_1=e^2\int_0^\infty\int_0^\pi f(r)f(r')r^2\sin\theta drd\theta,$$

where $$r'=\sqrt{r^2+z^2-2rz\cos\theta},$$

z being the distance between the electron-centres.

We can proceed otherwise as follows: The electric forces (PQR), $(P'Q'R')$ of the electrons are derived from potentials V, V', and the mutual energy in any region is

$$\frac{1}{4\pi}\iiint\left(\frac{\partial V}{\partial x}\frac{\partial V'}{\partial x}+\frac{\partial V}{\partial y}\frac{\partial V'}{\partial y}+\frac{\partial V}{\partial z}\frac{\partial V'}{\partial z}\right)d\tau$$

$$=-\frac{1}{4\pi}\iint V'\frac{\partial V}{\partial n}dS-\frac{1}{4\pi}\iiint V'\nabla^2 Vd\tau,$$

by the usual notation of Green's theorem, and the surface integral is negligible over the infinite boundary of the whole field, where V, V' are each of order $1/r$. Moreover, $\nabla^2 V'=4\pi\rho'$, and in the usual way,

$$W_1=\iiint V\rho' d\tau,$$

where the integral is taken over all space. This becomes

$$W_1=\frac{e\lambda^3}{8\pi}\iiint V\varepsilon^{-\lambda r'}r^2 drd\omega$$

$$=\tfrac{1}{4}e\lambda^3\int_0^\infty\int_0^\pi Vr^2\sin\theta\varepsilon^{-\lambda r'}drd\theta,$$

the origin being at the first electron, and $d\omega$ an element of solid angle. In order to obtain V, we note that it vanishes at infinity, and that $-\partial V/\partial r$ is the electric intensity due to an electron situated at the origin. Thus

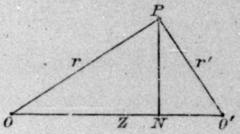
$$\frac{-\partial V}{\partial r}=\frac{e}{r^2}-e\left(\frac{1}{r^2}+\frac{\lambda}{r}+\frac{\lambda^2}{2}\right)\varepsilon^{-\lambda r}$$

$$V=\int_r^\infty \frac{edr}{r^2}-\int_r^\infty e\left(\frac{1}{r^2}+\frac{\lambda}{r}+\frac{\lambda^2}{2}\right)\varepsilon^{-\lambda r}dr$$

$$=\frac{e}{r}-\frac{e}{r}\left(1+\frac{\lambda r}{2}\right)\varepsilon^{-\lambda r}.$$

The potential at the centre of the electron is finite and equal to $\frac{1}{2}e\lambda$.

It is more convenient to use bipolar co-ordinates, for we must avoid, in the integrations, attaching a negative sign to r or r'.



Taking the centres O, O' of the electrons as origins, then any point, P, or (r, r') can equally be defined by the values of $r\pm r'$. Moreover, the curves $r\pm r'=$constant are the set of confocal ellipses and hyperbolas in any plane through O, O', with these points as foci. If $c=\frac{1}{2}z$, and $(\xi\eta)$ are the elliptic co-ordinates defined by

$$x=c\cosh\xi\cos\eta, \qquad y=c\sinh\xi\sin\eta,$$

then we readily find

$$(r, r')=c(\cosh\xi\pm\cos\eta),$$

which are essentially positive, since the minimum value of $\cosh\xi$ is unity. The semi-major axis of the ellipse through any point is, of course, $c\cosh\xi$, or $\frac{r+r'}{2}$.

Write $r+r'=\rho$, $r-r'=\sigma$, when r is greater than r'. Then ρ and σ are orthogonal co-ordinates in space, and their directions of increase are those of (ξ, η), or normal to the ellipse and hyperbola through any point in the plane of O, O'. The elements ds_1 and ds_2 of length along ρ and σ increasing are then given by

$$ds_1 = d\xi\left\{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2\right\}^{\frac{1}{2}} = c\sqrt{\cosh^2\xi - \cos^2\eta}\; d\xi$$

$$ds_2 = d\eta\left\{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2\right\}^{\frac{1}{2}} = c\sqrt{\cosh^2\xi - \cos^2\eta}\; d\eta$$

or
$$ds_1 ds_2 = c^2 d\xi d\eta(\cosh^2\xi - \cos^2\eta).$$

The third element may be defined by a rotation $d\varphi$ of the line PN round the axis O, O', and has a length

$$ds_3 = PN d\varphi = c \sinh \xi \sin \eta d\varphi.$$

Thus the volume of a rectangular element of space at the point P is

$$d\tau = c^3 d\xi d\eta d\varphi \sinh \xi \sin \eta(\cosh^2 \xi - \cos^2 \eta).$$

But
$$2c \cosh \xi = r + r' = \rho,\; 2c \sinh \xi d\xi = d\rho$$
$$2c \cos \eta = r - r' = \sigma,\; 2c \sin \eta d\eta = -d\sigma,$$

so that
$$d\tau = \frac{\rho^2 - \sigma^2}{16c} d\varphi d\rho d\sigma,$$

and the mutual potential energy becomes

$$W_1 = \int\!\!\int\!\!\int (V\rho') \, . \, \frac{\rho^2 - \sigma^2}{8z} d\varphi d\rho d\sigma.$$

Taking account of both cases, $r < r'$, $r' < r$, we find that this formula gives W_1 when taken between the limits

$$\varphi = o \text{ to } 2\pi$$
$$\rho = z \text{ to } \infty$$
$$\sigma = -z \text{ to } z.$$

Performing one integration

$$W_1 = \frac{\pi}{4z}\int_z^\infty d\rho \int_{-z}^{z} d\sigma(V\rho')(\rho^2 - \sigma^2)$$

in a form readily applicable to electrons of any law of density. For the present case

$$\rho = \frac{e\lambda^3}{8\pi}\varepsilon^{-\lambda r'} = \frac{e\lambda^3}{8\pi}\varepsilon^{-\frac{\lambda}{2}(\rho - \sigma)}$$

$$V = \frac{e}{r}\left\{1 - \left(1 + \frac{\lambda r}{2}\right)\varepsilon^{-\lambda r}\right\}$$

$$= \frac{2e}{\rho + \sigma}\left\{1 - \left[1 + \frac{\lambda}{4}(\rho + \sigma)\right]\varepsilon^{-\frac{\lambda}{2}(\rho + \sigma)}\right\},$$

and thus

$$W_1=\frac{e^2\lambda^3}{16z}\int_z^{\infty}\int_{-z}^{z}(\rho-\sigma)d\rho d\sigma\left\{1-\frac{\lambda(\rho+\sigma)}{4}\varepsilon^{-\frac{\lambda}{2}(\rho+\sigma)}\right\}\varepsilon^{-\frac{\lambda}{2}(\rho-\sigma)}$$

$$=\frac{e^2\lambda^3}{16z}(I_1+I_2),$$

where

$$I_1=\int_z^{\infty}\int^{z}d\rho d\sigma(\rho-\sigma)\varepsilon^{-\frac{\lambda}{2}(\rho-\sigma)}$$

$$I_2=-\frac{\lambda}{4}\int_z^{\infty}\int_{-z}^{z}d\rho d\sigma(\rho^2-\sigma^2)\varepsilon^{-\lambda\rho}.$$

Now

$$I_1=\varepsilon^{-\frac{\lambda z}{2}}\int_0^{\infty}\int_{-z}^{z}du d\sigma(z+u-\sigma)\varepsilon^{\frac{\lambda}{2}(\sigma-u)}\qquad(\rho=u+z)$$

$$=\varepsilon^{-\frac{\lambda z}{2}}\int_{-z}^{z}d\sigma\left\{\frac{2}{\lambda}(z-\sigma)+\frac{4}{\lambda^2}\right\}\varepsilon^{\frac{\lambda}{2}\sigma}$$

$$=\frac{2}{\lambda}\varepsilon^{-\frac{\lambda z}{2}}\left[\left\{\left(z-\sigma+\frac{2}{\lambda}\right)\frac{2}{\lambda}+\frac{4}{\lambda^2}\right\}\varepsilon^{\frac{\lambda}{2}\sigma}\right]_{-z}^{z}$$

$$=\frac{16}{\lambda^3}-\frac{4}{\lambda^2}\left(2z+\frac{4}{\lambda}\right)\varepsilon^{-\lambda z}$$

$$I_2=-\frac{\lambda}{2}\int_z^{\infty}d\rho\left(\rho^2z-\frac{z^3}{3}\right)\varepsilon^{-\lambda\rho}$$

$$=-\frac{\lambda}{2}\left[\left(\frac{z^3}{3\lambda}-\frac{\rho^2z}{\lambda}-\frac{2\rho z}{\lambda^2}-\frac{2z}{\lambda^3}\right)\varepsilon^{-\lambda\rho}\right]_z^{\infty}$$

$$=-\frac{\lambda}{2}\left(\frac{2z}{3\lambda}+\frac{2z^2}{\lambda^2}+\frac{2z}{\lambda^3}\right)\varepsilon^{-\lambda z}$$

$$=-\frac{z}{\lambda^2}\left(1+\lambda z+\frac{\lambda^2z^2}{3}\right)\varepsilon^{-\lambda z}.$$

And finally

$$W_1=\frac{e^2}{z}-\frac{e^2}{4z}(4+2\lambda z)\varepsilon^{-\lambda z}-\frac{e^2\lambda}{16}\left(1+\lambda z+\frac{\lambda^2z^2}{3}\right)\varepsilon^{-\lambda z}$$

is the mutual potential energy of the electrons, as a function of their distance z apart.

The force tending to increase z is $F=-\partial W_1/\partial z$, or

$$F=\frac{e^2}{z^2}-\frac{e^2\lambda^3z}{48}(1+\lambda z)\varepsilon^{-\lambda z}-\frac{e^2}{z^2}\left(1+\lambda z+\frac{1}{2}\lambda^2z^2\right)\varepsilon^{-\lambda z}$$

becoming e^2/z^2 at a great distance, as in the ordinary electron.

But at small distances the force does not tend to infinity, but to zero. We find

$$F\varepsilon^{\lambda z}=\frac{e^2}{z^2}\left(\varepsilon^{\lambda z}-1-\lambda z-\frac{1}{2}\lambda^2z^2\right)-\frac{e^2\lambda^3z}{48}\,(1+\lambda z)$$

$$=7\frac{e^2\lambda^3z}{48}\left(1+\frac{\lambda z}{7}\right)$$

when z is small in comparison with λ^{-1}. The force vanishes when the centres coincide, forming an electron of strength $2e$, as we should expect.

The exponentials are, for ordinary purposes, negligible even when z is a few multiples of λ^{-1}, which we may regard as the "radius" of the electron.

The electron $2e$ is essentially unstable, and e itself must be regarded as a constant of the aethereal structure, like λ. There is a probable relation between e and Planck's unit, as stated already.

Even when z is comparable with λ^{-1}, the force sinks to a small fraction of its value as given by the usual formula. For example, if $z=2/\lambda$, the force is only $0{\cdot}05e^2/z^2$.

This tendency of the force to vanish when the electrons are very close is, of course, applicable to electrons of opposite sign. But the inertia of the positive electron must be large, and this involves a large value of λ. We can show without difficulty that the inertia is proportional to λe^2 for slow speeds, and thus λ for the positive electron, if such exists, must be of the order of 1,000 times the value of λ for a negative electron.

But the practical evanescence of the force will remain, and there is even the possibility that the sign of the force may change, with a suitable law of density. Thus a positive and negative electron would not necessarily rush together and annihilate each other, but would form a doublet, whose length would be comparable with the radius of a single electron. Even if the force did not change sign, so that the positive and negative elements had no position of relative equilibrium, an oscillation about each other in simple harmonic motion is possible. For example, if the values of λ were equal, the electrons being of the above type, the equation for the distance z between them would be

$$\frac{m\ddot{z}}{2}=-\frac{7e^2\lambda^3z}{48},$$

where m is the mass of either. Since m is of order λe^2, the system would emit a wave-length comparable with the "radius" of an electron.

If the doublet is endowed with a rotation, it can preserve a constant length, and the present investigation is given merely as an illustration of the possibility of such doublets.

The view that two aethereal structures can exist in this way without deformation in presence of each other, and simultaneously occupying the whole aether, is, of course, difficult. But the difficulty is no greater than that of postulating the ordinary bounded electron, each of whose parts must repel each other. If actual deformation takes place, it is not apparently possible to find a basis of calculation without further hypotheses incapable of verification, so that the present suggestion of the possible nature and existence of doublets is sufficient for the purpose in view. It is, perhaps, worthy of remark that the Lorentz formula for mass as a function of velocity can be obtained for this type of electron, with λ^{-1} substituted for the radius. The whole distribution of density ρ may be treated as in the principle of relativity. It does not seem necessary to give the complete analysis.

From this point of view a neutral doublet could consist of a distribution of density of the form

$$\rho=\frac{e}{8\pi}(\lambda_1^3\varepsilon^{-\lambda_1 r}-\lambda_2^3\varepsilon^{-\lambda_2 r})$$

around the origin as centre. The total charge is zero when integrated throughout space, and the density and electric force are confined to a region round the origin whose radius is of order λ_1^{-1} $(\lambda_1<\lambda_2)$ almost precisely. The system behaves like a pair of charges $\pm e$ at the origin, of radii λ_1^{-1}, λ_2^{-1}, which can be separated by the application in a suitable form of an amount of energy of order $\lambda_1 e^2$.

Such a type of doublet, whose charges were any multiple of that of an electron, might form a component of a complex atomic nucleus, and radioactivity would then occur when it was partially dissolved into component α and β particles. It would appear to be necessary to consider a theory of this type, in view of the extreme smallness of the nuclei of atoms, as determined by Rutherford and others, which nevertheless possess the property of evolving large numbers of α and β particles, the latter being of the same order of size as the nucleus itself.

In again laying emphasis on the fact that the present note is only intended to be a suggestion towards a point of view, rather than a development of the view which could serve no useful purpose at the present stage—Sir Ernest Rutherford has proposed to leave nuclear structure to the next generation—it is to be noticed that the difficulties inherent in the permanent fixation of these types of strain, capable of superposition, are not greater than those involved in the internal equilibrium of the mutually repelling parts of the ordinary bounded electron.

ABSTRACT.

The electron is usually regarded as a globule of electricity with a definite radius. This conception has proved valuable, but involves difficulties in connection with the nuclear structure of complex atoms. On the view that the electron consists of a region of strain in the æther such line constants should have some significance throughout the whole æther; which may, in fact, be in some manner cellular with these linear magnitudes involved in the specification of the cells, and therefore in any strained structure composed of them.

The electron would be regarded as a state of strain which for practical purposes is concentrated at its centre, rapidly diminishing outwards according to some very convergent law involving some line constant in its specification. By way of illustration the idea is worked out mathematically on the assumption that the strain varies as $e^{-\lambda r}$, on which hypothesis λ^{-1} is the "radius." It can be shown that the Lorentz formula for mass as a function of velocity can be obtained for this type of electron. The charge on the electron is regarded as a fundamental property of the æther, and is related to Planck's constant h.

DISCUSSION.

Dr. H. S. Allen: There can be little doubt of the existence of a relation, referred to by Prof. Nicholson, between Planck's constant h and the charge of an electron e. The relation suggested by Lewis and Adams may be written—

$$ch = \sqrt[3]{\frac{8\pi^5}{15}(4\pi e)^2},$$

where c is the velocity of light. Taking Millikan's latest value for e $(4{\cdot}774 \times 10^{-10})$ and $c = 3 \times 10^{10}$, we find $h = 6{\cdot}558 \times 10^{-27}$. From his photoelectric experiments Millikan found $h = 6{\cdot}57 \times 10^{-27}$ within about 0·5 per cent., and in his latest table of fundamental constants he gives $h = (6{\cdot}547 \pm 0{\cdot}013) \times 10^{-27}$. Thus the agreement is within the limits of experimental error. All the principal radiation constants can be expressed in terms of e. The curious numerical relations between the primary constants of physics, to which attention was directed in my Paper read before the Society in 1915, depend upon the above formula connecting h and e. On the lines suggested in Prof. Nicholson's Paper it would seem as if most, if not all, of the important constants of nature may be referred to some fundamental property of the æther.

Sir Oliver Lodge (communicated): I am much interested in Prof. Nicholson's ingenious plan for doing away with the definite boundary of an electron, and devising a mathematical scheme which shall enable us to regard it as a point-centre of strain decreasing exponentially in every direction without limit, so that the linear dimension associated with it shall be—like many time-constants—the distance at which the density is reduced to $\frac{1}{e}$th of what it is at the centre. This plan, if it can be developed properly, seems to get over many of the difficulties about the coherence of parts of a charge, and about the extraordinary properties of a nucleus, which though, from some points of view, an extremely small and highly-charged unit, yet necessarily has a complexity which enables it to be disintegrated and fired off in fragments. The ready permeability or inter-penetrability without destruction, of Prof. Nicholson's conception of an electric unit, seems likely to diminish the difficulty of conceiving such a nucleus; and on the whole his suggestions seem to me helpful and valuable. I do not feel justified in saying more at the present time.

Prof. NICHOLSON, in reply, said it was of interest to see that Millikan's final value of h was practically equal to the first value that had been obtained for that constant.

II. *On the Thermo-Electric Properties of Fused Metals.* By CHARLES ROBERT DARLING *and* ARTHUR W. GRACE.

RECEIVED, OCTOBER 25, 1917.

METALS WITH MELTING POINTS BELOW 700°C.

IN a previous Paper on this subject ("Proceedings," Vol. XXIX., Part I.) an account was given of the experimental methods employed in investigating the behaviour of bismuth up to 560°C., which was the highest temperature attainable with the apparatus described. As the object of the research was to test the possibility of using a couple of one or two fused metals for measuring temperatures, it was necessary to resort to a new method of procedure in order to carry the observations with metals in general into the region above 600°C. to as high a limit as possible. The present Paper deals with the new experimental methods, and the results obtained with a number of metals up to 1,000°C.

Experimental.

Preliminary trials were made with a silica tube, of about 2·5 cm. inner diameter, closed at one end, and wound externally with a nichrom wire through which a current was passed. This heating element was lagged with magnesia, and the metal under test dropped into the tube and melted in situ. A wire of the second metal composing the couple, protected by a graphite tip, was inserted in the liquid mass, and side-by-side with this a pyrometer was placed. An overflow of the fused metal along a sheet of uralite was arranged as described in the previous Paper, the cold junction being formed at the end of the overflow. It was found, however, that after one or two heatings the silica tube was cracked, and as a second trial ended similarly, the method was abandoned in favour of others to be described, and illustrated in Figs. 1 and 2. In the first of these methods (Fig. 1) a silica tube *B*, open at both ends, was inserted about halfway up the vertical tube *A* of an electric furnace (nichrom wound), whilst the lower end of *B* dipped into a vessel containing oil. Prior to insertion in the furnace, *B* was filled with the metal under trial, and the second metal placed in the liquid mass in the form of wires, forming the hot junction *H* and the cold junction *C*. After placing in the furnace the upper part of the metal in *B* was melted, the pyrometer *P* inserted, and the hot and cold junctions coupled to a calibrated galvanometer *G*. Readings of E.M.F. were taken at suitable temperature intervals, the furnace being controlled by an external resistance. This method enabled satisfactory readings to be obtained with antimony, which expands on solidification, as the top surface remained liquid while solidification was proceeding below, thus permitting of free expansion. The second method was found to be more convenient than the foregoing in the case of metals procurable in the form of long rods. A graphite block, *G*, Fig. 2, was bored with two holes, into which were inserted tight-fitting silica tubes, *A* and *B*, 40 cm. long, each containing slack-fitting rods of the metals under examination. When one of the metals was known not to fuse at the temperature attained in the experiment the end was threaded and screwed into the graphite. This element was inserted in the tube of an electric furnace and used in the horizontal position, wires of the same material being taken from the cold end to the measuring apparatus. When placed in the furnace to a distance indicated by the line *D*, continuity of the circuit was maintained between the liquid portion in the furnace and the solid part beyond *D*.

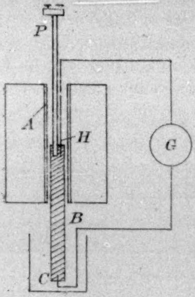


Fig. 1.

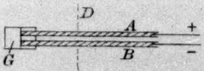


Fig. 2.

Temperatures were measured by means of a thermo-couple of Hoskins' alloys, kindly provided by the Foster Instrument Co. This couple was inserted in a cavity in the graphite block G, and in taking readings the furnace resistance was adjusted at intervals, and observations made when the pyrometer indicator and also the indicator connected to the metals under test were both stationary. The results obtained with a number of metals are appended, and refer to a cold-junction temperature of 25°.

Lead.

A large number of observations were made with lead, with a view to testing the validity of extrapolation in the thermoelectric diagram as well as to discover the effect of fusion. It was found that no abrupt change is produced on melting, all the curves obtained by plotting E.M.F. against temperature being quite smooth in this region. It was also found that whereas up to about 300°C. all these curves were approximately parabolic, the continuations above this temperature departed considerably from this shape. As an example of this, reference may be made to Fig. 3, which represents the values obtained with a german-silver-lead couple. There is no discontinuity at 327°, the melting point of lead, but from about 300° upwards the temperature—E.M.F. relation becomes linear, and consequently $\frac{dE}{dT}$ has a constant value over this range. The representation of this couple on a thermo-electric diagram would, therefore, consist of a sloping line up to 300°, and afterwards of a horizontal line parallel to the lead axis. It is evident,

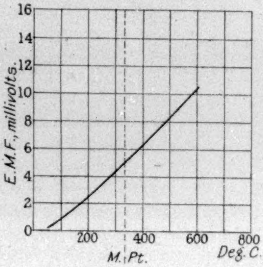


FIG. 3.

therefore, that extrapolation of the sloping lines obtained from low-temperature observations leads to serious errors, and many published diagrams are quite erroneous for this reason. As an example, it is specifically stated in some books that the neutral point of iron and lead is 350°C. beyond which the E.M.F. diminishes; whereas direct observation shows that no such neutral point exists, and that the E.M.F., which is 2·6 mv. at 350°, rises continuously to a value of 4·2 mv. at 900°. A correct thermo-electric diagram for various metals up to 1,000°C. has yet to be prepared; and as such would no longer possess the simple character of one based on a parabolic relation between E.M.F. and T, it would be of doubtful value.

Tin.

As the high boiling point of tin renders it feasible for use as one of the members of a liquid junction, many experiments have been made with this end in view. In no instance was any discontinuity noticed at the melting point (232°C.), as indicated in Fig. 4 (iron-tin), in which the shape of the curve in this region is well suited to the detection of even a slight change. This curve also illustrates the dangers of extrapolation, based on the usual assumption that the curve beyond the neutral point will be a geometric continuation of the earlier portion. As will be seen, the steepness diminishes considerably after passing the neutral point and between 700° and 850° shows a flexure, after which the steepness again increases. This behaviour of iron in the recalescence region was first noticed by Belloc,[*] and is well shown when coupled with tin,

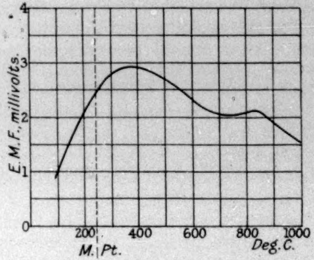


FIG. 4.

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[*] Ann. de Chim. et de Phys., 30, p. 42, 1903.

although obscured in the case of an iron-constantan junction, when the E.M.F. under measurement is 20 times as large, entailing the use of a much coarser indicator.

Cadmium, Zinc and Aluminium.

In the case of these three metals it was also noticed that the act of melting produced no change in the E.M.F. The temperature of inversion of zinc and iron was observed to be about 470°, which is about 50° above the melting point, and had any change resulted from fusion it would have been detected readily with this couple. Aluminium and constantan show a linear relation between E.M.F. and temperature, which is not interrupted by fusion at 657°.

Antimony.

Special interest was attached to the experiments with antimony, owing to the general resemblances between this metal and bismuth. After several unsatisfactory attempts with different arrangements, a successful set of readings were obtained with the apparatus shown in Fig. 1. It was found that, as in the case of bismuth, an abrupt change in thermo-electric properties occurred at the melting point, 632°. A typical case is shown in Fig. 5, which shows the behaviour of an antimony-copper couple between 400° and 1,000°, the change in shape due to fusion being most pronounced. A similar result was obtained with an antimony-iron couple, and hence antimony acts in the same manner as bismuth in this respect.

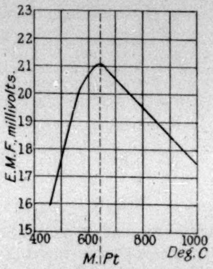

Fig. 5.

Conclusions.

So far as it is possible to generalise on the results obtained, it would appear that the thermo-electric properties of metals are usually unaffected by change of state from the solid to the liquid phase or vice versa. The exceptional behaviour of bismuth and antimony may be due to the formation of allotropic modifications on melting, in support of which view may be adduced the fact that both of these metals expand upon solidification, and are thus exceptions to the ordinary rule. Further, as shown in the case of iron, an allotropic change is accompanied by an alteration in thermo-electric properties, and it is possible that the one change is always accompanied by the other. Experiments are now in progress with metals of still higher melting points, which may confirm or otherwise the view expressed above. The success of the main object of the research—the production of a high-reading pyrometer—entails the condition that mere change of state has no effect on the thermo-electric properties of the metals used.

It is possible that molecular changes occurring in molten alloys may be detected by experimental methods similar to those described in the present communication. A suitable metal to couple with the alloy under test could be found by trial, and it is probable that the change in E.M.F. accompanying a molecular transformation would in some cases be detected with greater certainty than a small temperature halt. This is a matter which it is hoped to investigate later, as opportunities permit. Experiments are also desirable in which both metals forming the couple undergo liquefaction, which it is also hoped to conduct at some future time.

We would point out that the values of E.M.F. given in the present and the previous Paper refer only to our samples of metals, which were purchased without any specification as to purity. As is well known, different specimens of what are reputed to be the same metal frequently vary in thermo-electric properties; thus two pieces of platinum wire from different sources usually show an E.M.F. when joined and heated. We have, therefore, preferred in all cases to take a direct observation, rather than to work with a single metal and deduce the other results, as suggested by A. Campbell in a criticism of the previous Paper. We find that we have thus saved ourselves from many errors, particularly in the case of alloys such as constantan and nichrom, which vary considerably in composition, but are nevertheless very valuable in thermo-electric work.

ABSTRACT.

In a previous Paper ("Proceedings," Vol. XXIX., Part I.) the authors described experiments with bismuth, the apparatus then used only being capable of furnishing readings up to 560°C. Methods have now been devised in which the metals examined may be heated in the tube of an electric furnace, and observations made up to the temperature limit of the furnace. The metals experimented with were lead, tin and antimony up to 1,000°C., and zinc and cadmium up to temperatures approaching the boiling point. No change in thermoelectric properties was noticed at fusion, except in the case of antimony, which, like bismuth, shows an abrupt bend in the E.M.F.-temperature curve at the melting point, 632°C. This exceptional behaviour of antimony and bismuth is in keeping with the anomalous properties of these metals, both of which expand on solidification; and it is suggested that an allotropic change occurs at fusion in these metals.

In the case of lead which is used as the reference metal in thermo-electric diagrams, it is shown that extrapolation of lines in the diagram beyond 300° led to serious errors, and that although at low temperatures the E.M.F.-temperature curves are approximate parabolas, the departure from this shape above 300° is so marked as to render thermo-electric diagrams of little value.

DISCUSSION.

Mr. **Whipple** said that this work opened up certain possibilities of commercial importance, as it appeared that information could be obtained of the thermo-electric properties which a particular alloy would have while it was still in the molten state. It would thus be possible by adding one or other of the constituents as required to obtain an alloy with any prescribed thermo-electric properties. At present the alloy had to be allowed to cool, and a wire of it drawn and tested. If it were not right, it had to be melted up again and its constitution altered, which was a troublesome method. With regard to the high boiling point and low vapour pressure of tin, it was of interest to observe that Northrup had suggested a tin graphite thermometer for high temperatures up to about 1,700°C. on the same lines as the ordinary mercury in glass thermometers. The tin expanding along the stem of the thermometer moved an index wire by which the temperature was indicated. What was the magnitude of the change in properties of the iron-constantan couple on passing the recalescence point of iron? He was very interested in this point, as he had not noticed any such change himself.

The **President** said it was a useful thing to have the futility of the old thermo-electric diagram proved so thoroughly.

Dr. **Willows** suggested that a zinc-mercury amalgam would be an interesting substance to examine thermo-electrically for allotropic change-points. Its resistance curve between 0° and 100° shows marked evidence of such changes.

Mr. **Darling**, in reply, said he had not detected the recalescence change with the iron-constantan couple probably because of the relatively large total E.M.F. of that couple. The change was easily detected, however, with the iron-tin couple, since the total E.M.F. is then only about 3 millivolts and a delicate galvanometer is used.

III. *Triple Cemented Telescope Objectives.* *By* T. SMITH, *B.A., and* Miss A. B. DALE. (*From the National Physical Laboratory.*)

RECEIVED OCTOBER 18, 1917.

THE ordinary telescope objective consists of a crown lens and a flint lens which are not cemented together because the conditions which must usually be satisfied demand a difference in the curvatures of the inner surfaces of the two lenses. For some purposes it is desirable to have a cemented objective, even if this involves some falling off in the quality of the definition obtainable. There are, however, limits beyond which such deterioration may not go, and these depend on the relative aperture and the field of view of the lens. Unfortunately the circumstances in which a cemented objective is required are usually those which also involve large relative apertures and a large field of view. If a triple cemented objective is substituted for a doublet the extra degree of freedom obtained enables the required conditions to be satisfied much more nearly than in the case of the simpler form of objective, even though no additional variety of glass is used for the third component lens. The extent of the advantage thus gained depends upon the magnitude of the outstanding aberrations and upon the magnitude of the curvatures required for the triple objective. A manufacturer will require to have shallower surfaces to grind as a compensation for the additional labour involved in the introduction of two extra surfaces into the optical system, and the performance of the telescope will not be sufficiently improved if the reconciliation of the conditions for the removal of spherical aberration and coma of the first order involves the introduction of greatly increased aberrations of the second order. The object of the investigation described in the present paper is to determine what two kinds of glass should be used in order to obtain a triple telescope objective that is most satisfactory as regards smallness both of curvatures and of second order aberrations.

The commonly accepted opinion is that these two characteristics go hand in hand, so that among otherwise equally good objectives that one will have the smallest second order aberrations which has the least curvatures. This, no doubt, may be taken as a rough guide, but it will be seen from the results detailed below that it is not strictly true, for the series of triple objectives which are best as regards smallness of second order aberrations have somewhat greater curvatures than another series of triplets which are not quite so satisfactory in this respect.

In the calculation of all objectives thicknesses have been entirely neglected. This course enables the amount of numerical work involved to be reduced to the minimum without causing errors of any consequence in the conclusions drawn from the results of the investigation. A further simplification has been introduced by assuming the refractive index of the flint glass to be 1·6200 throughout the whole series of objectives. With this one exception the results are quite general, as both the refractive index of the crown glass and the ratio of the dispersions produced by the two glasses have been varied over a sufficiently wide range to embrace all the varieties of glass that are likely to be employed for the construction of telescopic objectives.

The objectives considered have in all cases been calculated so that all first order spherical aberration and coma is removed. These are not the conditions that are satisfied in actual objectives, a balance between first and second order aberrations of opposite signs being decidedly preferable. It might be supposed that this would detract from the utility of the investigations; this is, however, not the case. The curvatures of the actual objectives in which such a balance is obtained are almost identical with those of thin objectives free from first order aberrations, and the differences in the magnitude of the second order aberrations must, therefore, be small if the actual objectives are of normal thickness.

The formulæ from which triple objectives are to be calculated have been given in a previous paper by one of the authors.* The triplet is most simply regarded as two doublets cemented together; the aberrations of a triplet then depend upon :—

(*a*) The minimum spherical aberration for magnification −1 for the doublet, the corresponding amount of coma, and the curvatures of the external surfaces;

(*b*) The amount of spherical aberration when a plane wave is incident normally on a doublet having its first surface plane;

(*c*) The Petzval sum.

These quantities are plotted in Figs. 1 to 4, for a varying refractive index of the crown glass and for ν ratios of crown to

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* "Notes on the Calculation of Thin Objectives." Proc. Phys. Soc., Vol. XXVII., p. 495.

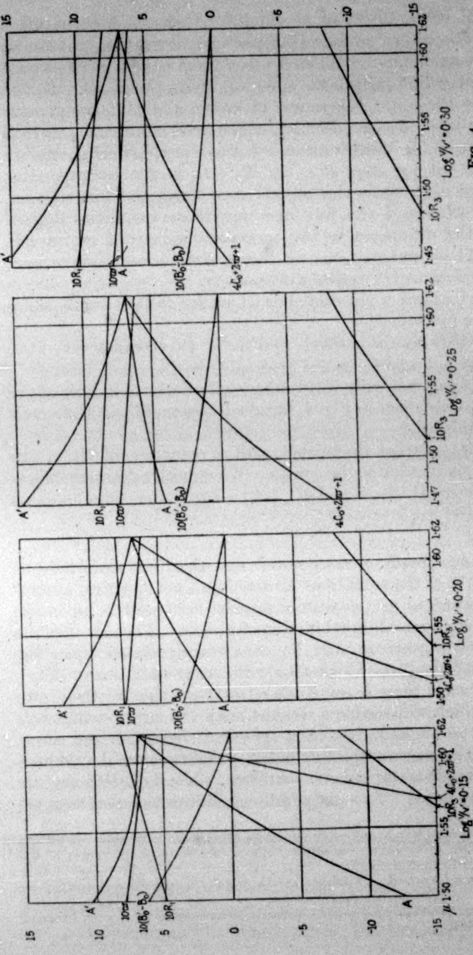

FIG. 1.

FIG. 2.

FIG. 3.

FIG. 4.

FUNDAMENTAL CONSTANTS OF DOUBLETS.

flint of $10^{0\cdot15}$, $10^{0\cdot20}$, $10^{0\cdot25}$ and $10^{0\cdot30}$.* For a doublet with a flat outer crown surface the spherical aberration is A for an object at infinity. When a plane wave is first incident on a flat outer flint surface the spherical aberration is A'. R_1 and R_3 are the outer curvatures of crown and flint components respectively of the doublet which has minimum spherical aberration for magnification -1, and this spherical aberration is denoted by $4C_0+2\varpi+1$. $B_0'-B_0$ is the corresponding amount of coma in this objective, and ϖ is the Petzval sum. A knowledge of the way in which these quantities depend upon the difference in the refractive indices of crown and flint glasses and upon their ν ratio is of the greatest importance to the designer of optical systems.

The features of the diagrams which are of most consequence are the following :—

(*a*) A' is always positive, and the A' curve varies very little in Figs. 2, 3 and 4.

(*b*) A is practically constant for all refractive indices of the crown component in Fig. 4, but decreases rapidly as the ν ratio is diminished.†

(*c*) $4C_0+2\varpi+1$ diminishes with a reduction of either the crown glass index or the ν ratio. When both refractive indices become equal, A, A' and $4C_0+2\varpi+1$ take the common value $\left(\frac{\mu}{\mu-1}\right)^2$.

It has already been pointed out that the simultaneous fulfilment of the conditions for freedom from spherical aberration and coma in a cemented doublet is dependent upon the choice of a suitable combination of glasses. The classification of triple objectives may be conveniently based upon the cemented doublets which satisfy the same conditions. For a given ν ratio there is some one refractive index for the crown glass which will enable a doublet with the crown component leading to be made free from spherical aberration and coma. If now the crown refractive index is slightly altered a doublet can be made to satisfy one condition, while departing slightly from the other. To satisfy both conditions the doublet must be replaced by a triplet, and obviously this triplet will only depart slightly from the doublet form, the additional component being a very weak lens. Evidently two triplets can be found to satisfy the conditions, one consisting of a weak crown component added after the flint lens, and another with a weak flint component placed in front of the crown lens. As greater variations in the refractive index are introduced the departures from the doublet form will become more pronounced, but over an appreciable range of indices it is to be expected that the derivation of these two triple objectives from a doublet with the crown component leading will be evident. In a like manner two series of triple objectives may be derived from a doublet with the flint component leading. The classification adopted here will therefore be as follows :--

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* If the objective is not to be corrected exactly for chromatic aberration these figures represent the absolute value of the ratio of the power of the crown lens to that of the flint.

† Glasses for which A vanishes are those suitable for the construction of symmetrical triplets for magnification -1. Since A' is always positive there is no possibility of a corresponding symmetrical form with external crown lenses.

Series I. and II.—External Lenses of Crown Glass.

Series I. based on doublet with crown component leading, and therefore tending to have the first crown component more powerful than the second.

Series II. based on doublet with flint component leading, and therefore tending to have the first crown component weaker than the second.

Series III. and IV.—External Lenses of Flint Glass.

Series III. based on doublet with crown component leading, and therefore tending to have the front flint component weaker than the second.

Series IV. based on doublet with flint component leading, and therefore tending to have the front flint component more powerful than the second.

The curves showing any properties of Series I. and III. must necessarily intersect at the position relating to the doublet, since this is a member of both series. Similarly, curves relating to II. and IV. must intersect at the position of the second doublet. It is to be noted that there is no reason why the curves showing certain properties of these allied series should not also intersect at additional points.

Each series is divided by the double points into two parts, and the character of the triplet in these sections is quite distinct. The distribution of the total power of the lenses made of the external glass between the two components varies progressively in the series, and at the double points the power of one of these components becomes zero. If, then, the powers of both external components are of the same sign on one side of the double point they will necessarily be of opposite sign on the other side. It is to be expected that the curvatures and higher order aberrations of any one series will be greater in absolute value on the latter side of the double point. This side, it will be seen, is that on which the difference of refractive indices of the two glasses is less than for the doublet. From results previously established for double objectives * it follows that the normal glasses now available lie in the region more favourable for small curvatures and higher order aberrations.

If in any region the solution for a series becomes imaginary, the solution for another series must also have the same property. The series associated in this way with one another are obviously those which have the same glass externally, since in such a case the one series must be a continuation of the other. It is found that the solutions to Series I. and II. are always real, but those of Series III. and IV. become imaginary when the difference between the refractive indices exceeds a certain amount depending upon the ν ratio.

Figs. 5, 6, 7 and 8 show the second order aberration coefficients for triple objectives calculated from the data given in Figs. 1 to 4. The full curves show the amount of spherical aberration and the dotted curves the departure from the sine condition. The heavy lines relate to Series I. and III., which approximate to a doublet with the crown component leading, and the lighter lines to Series II. and IV. It will be noted that these two outstanding quantities are always negative. Thus the spherical aberration of the second order always tends in the direction of over-correction. The coma depends upon the difference between the full and the dotted curves, and is thus of one sign for Series I. and III. and of the opposite sign for Series II. and IV. Generally speaking, the amount of coma does not vary very greatly between the various series except near the junction of Series III. and IV., where the coma is small for systems with external flint components. As would be expected from a knowledge of the second order aberrations of achromatic doublets, the spherical aberration is distinctly less for Series I. than for Series II., and for Series III. than for Series IV. The most interesting feature of these diagrams, however, is the way in which the curves for combinations with external flint lenses lie above those with external crown lenses,

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* *See* "The Choice of Glass for Cemented Objectives." Proc. Phys. Soc., Vol. XXVIII., Figs. 1 and 2.

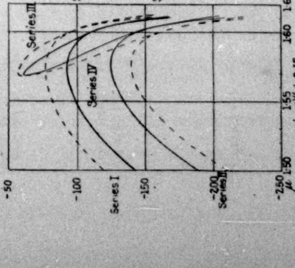

Fig. 5.

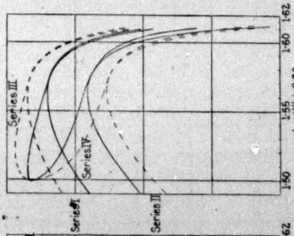

Fig. 6.

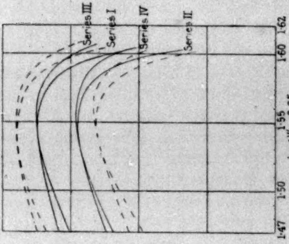

Fig. 7.

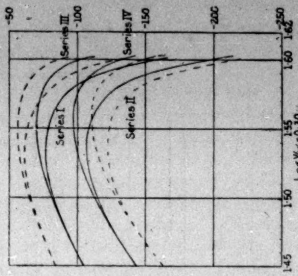

Fig. 8.

Second Order Spherical Aberration and Coma.

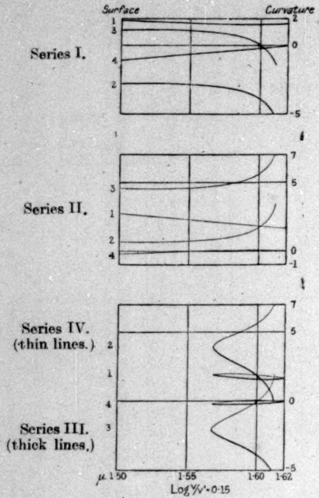

FIG. 9.

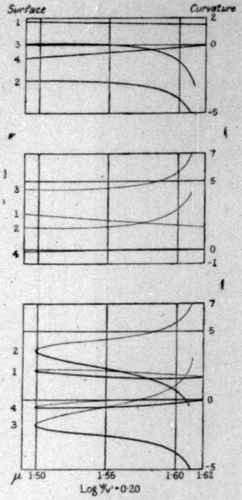

FIG. 10.

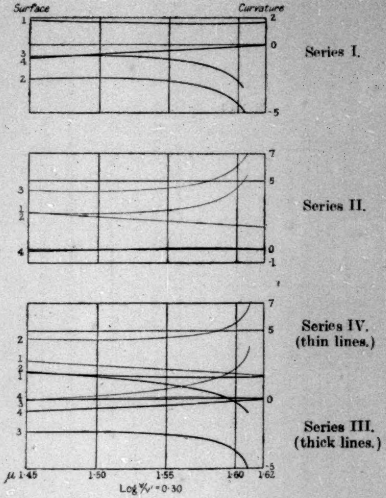

FIG. 11.

CURVATURES OF TRIPLET SURFACES.

Series III. thus being the best as regards spherical aberration. It is also remarkable that the minimum amount of this aberration left outstanding should vary so slightly over the range of ν values covered by the diagrams.

It was originally intended, when this investigation was undertaken, to plot in a diagram, in which the refractive index of the crown glass and the ν ratio were the abscissæ and ordinates respectively, contours showing for which glasses the aberrations attained equal values. The four figures shown, however, indicate that this would involve an amount of labour which the results would not justify. For the contours are not in all cases simple closed curves, but are broken up into at least two parts, as may be seen by noting that in Figs. 5, 6 and 8 the curves cross the line -60, but do not reach it in Fig. 7. The approximate constancy of the position of the highest points of the curves removes the need for accurately plotted contours of the character at first proposed.

The remaining figures show the curvatures of the surfaces. The curvatures of the external surfaces vary between comparatively close limits, and are chiefly determined by the condition for freedom from coma. The curves giving the curvatures of the two cemented surfaces are approximately parallel to one another. The inner surfaces tend to become very strongly curved when the refractive index of the crown lens approaches that of the flint. In Series I. and III., represented by the thick curves, the greatest curvatures are distinctly less than for Series II. and IV., shown by thin lines. In the former two series the cemented surfaces become concave to the incident light, and in the latter two series convex, when the refractive index of the crown glass is only slightly less than that of the flint. The curvature of the steepest surface in Series I. is sometimes greater and sometimes less than in Series III.; but the curvatures of Series I. are decidedly the more favourable when the outstanding aberration of Series III. is least. In the case of greatest practical importance the two conditions are therefore inconsistent. The triple objectives of both Series I. and III. show decided advantages over most doublets, both as regards smallness of curvature and smallness of outstanding spherical aberration. Considerable gain is thus possible by their use for many purposes where the conditions are distinctly difficult of fulfilment with an objective of the ordinary type.

Figs. 9, 10, 11 suggest that for small variations in the μ and ν values of the glasses the principal alterations in the curvatures should usually be carried out on the surfaces indicated in the following table :—

Series.	Variation in μ.	Variation in ν.
I.	4	3
II.	1	2
III.	2 and 3	2
IV.	3 and 2	3

In all cases a variation in ν is chiefly compensated by an alteration of an internal surface. In lenses with exterior crown components alterations for both μ and ν should be carried out on the weak correcting component. The fact that each correction can be carried out on a single surface constitutes an important manufacturing advantage for these two series. If Series I. is adopted, and the only factor likely to vary appreciably is the refractive index of the glass, when objectives of a given focal length are required it should be possible to use with all glass meltings the same standard tools for all the surfaces excepting the last.

ABSTRACT.

The Paper describes the four series of triple cemented thin telescope objectives which can be made from two kinds of glass, and determines their construction when first order spherical aberration and coma are eliminated. The second order spherical aberration and coma are then calculated, and the former found to be of the same sign for all optical glasses when the surfaces are spherical. The best standard attainable varies very little over a considerable range of glasses. Diagrams show the variations in the curvatures as the glasses are varied for refractive index and dispersion. Contrary to the general belief, it is found that the objectives with least second order aberrations (absolute values) are not those with the least curvatures for their refracting surfaces.

IV. *On a Class of Multiple Thin Objectives.* *By* T. Smith, B.A. (*From the National Physical Laboratory.*)

Telescope objectives usually consist of two component lenses mounted in contact, and their optical properties differ very little from those of infinitely thin lenses. An objective consisting of only a single piece of glass may be made free from spherical aberration and from coma for light of a specified wave-length, but its surfaces will in general not be spherical. By using two components made from glasses of suitably differing optical properties, the chief defect of the single lens, the large differences in the position of the focus with light of various colours, may be replaced by a much less serious fault. With a single lens the distance of the focus from the objective invariably increases as the wave-length of the light is increased. When two kinds of glass are employed this distance may be made first to decrease, then become stationary, and afterwards to increase; the rates of both increase and decrease under these conditions are many times less than the corresponding rate of increase in the single lens. The wave-length for which the position of the focus is stationary is carefully selected according to the purpose for which the objective is to be used. When this wave-length is given the relation between the positions of the focus for light of various wave-lengths is quite definite, and depends only upon the kinds of glass of which the objective is made. No matter how the shapes of the lenses may vary, all the objectives of the same two glasses and of the same focal length will have identically the same relation between the position of the focus and the wave-length of the light if the focus is stationary for the same wave-length in every case. It follows that any conditions relating to the chromatic aberration of the focal distance must be met by a suitable choice of the kinds of glass of which an objective is to be made, and that the shapes of the lenses may be arranged to satisfy other aberration conditions.

When two lenses are employed it is found that the two remaining conditions which it is most important to satisfy, freedom from spherical aberration and from coma for light of a given colour, can be satisfied, though the four surfaces are restricted to the spherical form. This being so, objectives have almost invariably been designed with spherical surfaces to the exclusion of all other forms, on account of the great advantages the former offer in ease of manufacture and testing in comparison with the latter. It is important to realise that the difficulties of manufacturing non-spherical surfaces to the extremely high degree of accuracy essential in optical work have been the decisive factor. The calculation of lenses having surfaces of other forms offers little difficulty. In fact, the calculation of such optical systems as photographic lenses, where many conditions must be approximately satisfied at the same time, would be very greatly simplified if the restriction to spherical surfaces were removed.

The statement that the removal of spherical aberration and coma from a doublet lens is consistent with the employment of spherical surfaces needs some qualification. It is strictly true if by spherical aberration and coma, first order spherical aberration and first order coma are meant. These are much the most important monochromatic aberrations in telescope objectives, but the corresponding aberrations of the second and third orders are appreciable, particularly in objectives of large relative aperture. The conditions for the removal of these residual aberrations are not compatible with the absence of the first order aberrations when the surfaces are strictly spherical, and such an objective will consequently bring the light which has traversed various zones of the lens aperture to somewhat different foci. In large telescope objectives the residual aberrations are reduced by deliberately departing from the spherical form for one or more of the lens surfaces. This " figuring " process, as it is called, is much too costly to be applicable to lenses that are required in large numbers, but it fortunately happens that the need of figuring in the case of these smaller lenses is comparatively rare. There are nevertheless many cases in which the removal of the zonal aberrations would lead to an appreciable improvement in the performance of the optical system to which the objective belongs.

In many instruments in which thin objectives are employed the total number of lenses is inevitably considerable, and it is important to avoid the introduction of unnecessary glass-air surfaces on account of the loss of light which each involves. Wherever possible the components of an objective should have equal curvatures on successive internal surfaces, so that these may be cemented together, and the losses by reflexion reduced. When the objective is a cemented doublet two aberration conditions can only be satisfied by making choice of the proper kinds of glass.* With a triplet using only two kinds of glass two aberration conditions can always be satisfied.† As a rule, four triplets can be found to satisfy the conditions. The cemented triplet has the same number of degrees of freedom and the same limitations as the uncemented doublet; that is to say, the satisfaction of the first order aberration conditions is inconsistent with the absence of higher order aberrations when the surfaces are spherical‡. One of the simplest ways in which the residual aberrations might possibly be removed consists in the employment of only two kinds of glass, but with at least two lenses of each glass, the condition that all the surfaces are to be spherical, being, of course, retained. The present paper deals with a method by which such multiple lenses can be calculated. A series of lenses consisting of four thin components has been calculated by the method described, and their second order spherical aberration coefficients determined. The number of lenses of this kind so far investigated is too small to enable the maximum advantages which may be secured to be estimated. The series examined, however, shows that the ratio of the shortest radius to the focal length may be very much greater than in simpler lenses satisfying the same first order conditions, and suggests that it may be possible to secure this advantage in a lens containing surfaces of only three different radii.

In addition to the defects already mentioned there is another which is often serious. When the spherical aberration has been corrected for light of one wave-length the spherical aberration for sensibly different wave-lengths is far from negligible. The method of calculation adopted enables the chromatic differences of first order spherical aberration and first order coma to be determined at once in terms of those of a cemented achromatic doublet of the same two glasses. It is not necessary for this purpose to complete the determination of the compound lens.

The method employed is an extension of that already adopted for the calculation of triple objectives. The triple objective is regarded as a combination of two cemented achromatic doublets, the external surfaces of one kind of glass being of equal and opposite curvatures, so that when the two doublets are put together no air lens is left between them. It may be supposed that n achromatic doublets are to be combined in the present instance. Let the typical doublet of unit focal length have its crown component of power K, and its flint of power K'. Then

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* "Notes on the Calculation of Thin Objectives." Proc. Phys. Soc., Vol. XXVII., p. 498.

† "The Choice of Glass for Cemented Objectives." Proc. Phys. Soc., Vol. XXVIII., p. 232.

‡ "Triple Cemented Telescope Objectives." Proc. Phys. Soc., Vol. XXX., p. 21.

$$K+K'=1.$$

Take n real numbers, $\kappa_1, \kappa_2, \kappa_3, \ldots \kappa_{n-1}, \kappa_n$, whose sum is $\kappa_{1,n}$. The system of n thin doublets of the standard type of powers $\kappa_1, \kappa_2, \kappa_3, \ldots \kappa_{n-1}, \kappa_n$, placed in contact will be achromatic and of power $\kappa_{1,n}$. If the odd numbers have their crown components leading and the even numbers their flint components leading, and the curvature of the last surface of the λth component is equal and opposite to the curvature of the first surface of the $(\lambda+1)$th component for all values of λ from 1 to $n-1$, the resulting system consists of $n+1$ lenses, the odd numbers being of crown glass and the even numbers of flint glass. The powers of these lenses are

$\kappa_1 K, (\kappa_1+\kappa_2)K', (\kappa_2+\kappa_3)K, (\kappa_3+\kappa_4)K', \ldots (\kappa_{n-1}+\kappa_n)K, \kappa_n K'$
if n is odd, and

$\kappa_1 K, (\kappa_1+\kappa_2)K', (\kappa_2+\kappa_3)K, (\kappa_3+\kappa_4)K', \ldots (\kappa_{n-1}+\kappa_n)K', \kappa_n K$
if n is even.

When n thin systems are combined to form a new thin system, the aberration coefficients of the latter may be calculated from those of the components by means of the equations

$$\kappa_{1,n}\varpi_{1,n}=\sum_1^n\kappa_\lambda\varpi_\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\kappa^2_{1,n}(B'_{1,n}-B_{1,n})=\sum_1^n\{\kappa_\lambda^2(B_\lambda'-B_\lambda)-(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda\varpi_\lambda\}. \quad (2)$$

and

$$\begin{aligned}\kappa^3_{1,n}&(2C_{1,n}+\varpi_{1,n})\\ &=\sum_1^n\{\kappa_\lambda^3(2C_\lambda+\varpi_\lambda)-2(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda^2(B_\lambda'-B_\lambda)\\ &\qquad\qquad+(\kappa_{1,\lambda}-\kappa_{\lambda,n})^2\kappa_\lambda\varpi_\lambda\}. \quad . \quad . \quad (3)\end{aligned}$$

The first equation follows immediately from the definition of ϖ. The other two may be verified by induction. They are known to hold for two thin systems in contact. Assuming that (2) holds for a definite value of n and for $n=2$, let the system 1 to n be taken as the first component in the latter case, and

$(n+1)$ as the second component. Thus, in addition to equation (2) the relation

$$\begin{aligned}\kappa^2_{1,n+1}(B'_{1,n+1}-B_{1,n+1})\\ &=\kappa^2_{1,n}(B'_{1,n}-B_{1,n})+\kappa^2_{n+1}(B'_{n+1}-B_{n+1})\\ &\qquad+\kappa_{n+1}\kappa_{1,n}\varpi_{1,n}-\kappa_{1,n}\kappa_{n+1}\varpi_{n+1}\end{aligned}$$

is given. Substitute in this equation from (2) for $\kappa^2_{1,n}(B'_{1,n}-B_{1,n})$. Then

$$\begin{aligned}\kappa^2_{1,n+1}(B'_{1,n+1}-B_{1,n+1})\\ &=\sum_1^n\{\kappa^2_\lambda(B_\lambda'-B_\lambda)-(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda\varpi_\lambda\}\\ &\qquad+\kappa^2_{n+1}(B'_{n+1}-B_{n+1})+\kappa_{n+1}\kappa_{1,n}\varpi_{1,n}-\kappa_{1,n}\kappa_{n+1}\varpi_{n+1}\\ &=\sum_1^{n+1}\{\kappa_\lambda^2(B'_\lambda-B_\lambda)-(\kappa_{1,\lambda}-\kappa_{\lambda,n+1})\kappa_\lambda\varpi_\lambda\}\\ &\qquad-\kappa_{n+1}\sum_1^n\kappa_\lambda\varpi_\lambda+\kappa_{n+1}\kappa_{1,n}\varpi_{1,n}\\ &=\sum_1^{n+1}\{\kappa_\lambda^2(B_\lambda'-B_\lambda)-(\kappa_{1,\lambda}-\kappa_{\lambda,n+1})\kappa_\lambda\varpi_\lambda\}\end{aligned}$$

by (1). It follows that (2) is always true for any compound thin lens.

Similarly from (3) and from the special case

$$\begin{aligned}\kappa^3_{1,n+1}(2C_{1,n+1}+\varpi_{1,n+1})\\ &=\kappa^3_{1,n}(2C_{1,n}+\varpi_{1,n})+\kappa^3_{n+1}(2C_{n+1}+\varpi_{n+1})\\ &\quad+2\kappa_{n+1}\kappa^2_{1,n}(B'_{1,n}-B_{1,n})-2\kappa_{1,n}\kappa^2_{n+1}(B'_{n+1}-B_{n+1})\\ &\quad+\kappa^2_{n+1}\kappa_{1,n}\varpi_{1,n}+\kappa^2_{1,n}\kappa_{n+1}\varpi_{n+1}\end{aligned}$$

it follows that

$$\begin{aligned}\kappa^3_{1,n+1}(2C_{1,n+1}+\varpi_{1,n+1})\\ &=\sum_1^{n+1}\{\kappa_\lambda^3(2C_\lambda+\varpi_\lambda)-2(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda^2(B_\lambda'-B_\lambda)\\ &\qquad\qquad+(\kappa_{1,\lambda}-\kappa_{\lambda,n+1})^2\kappa_\lambda\varpi_\lambda\}\\ &\quad-2\kappa_{n+1}\sum_1^n\kappa_\lambda^2(B_\lambda'-B_\lambda)+2\kappa_{n+1}\kappa^2_{1,n}(B'_{1,n}-B_{1,n})\\ &\quad-\kappa^2_{n+1}\sum_1^n\kappa_\lambda\varpi_\lambda+2\kappa_{n+1}\sum_1^n(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda\varpi_\lambda+\kappa^2_{n+1}\kappa_{1,n}\varpi_{1,n}\end{aligned}$$

or by (1) and (2)

$$\begin{aligned}\kappa^3_{1,n+1}(2C_{1,n+1}+\varpi_{1,n+1})\\ &=\sum_1^{n+1}\{\kappa_\lambda^3(2C_\lambda+\varpi_\lambda)-2(\kappa_{1,\lambda}-\kappa_{\lambda,n+1})\kappa_\lambda^2(B_\lambda'-B_\lambda)\\ &\qquad\qquad+(\kappa_{1,\lambda}-\kappa_{\lambda,n+1})^2\kappa_\lambda\varpi_\lambda\}\end{aligned}$$

which is the result required for inductive verification.

Let σ_λ denote the value of $\Sigma\kappa R$ for a lens similar to the λth component, but of unit power. The corresponding quantity for the λth component will be $\kappa_\lambda{}^2\sigma_\lambda$, since each factor will be altered in the ratio $\kappa_\lambda : 1$. For the complete system the sum is therefore given by

$$\kappa^2{}_{1,n}\sigma_{1,n}=\Sigma\kappa_\lambda{}^2\sigma_\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

There are $n-1$ geometrical conditions to be satisfied, if for all values of λ the last surface of the λth component is to be equal in curvature to the first surface of the $(\lambda+1)$th component, so that the two may be cemented together. If the curvature of each surface of the λth component is greater by r_λ than when C_λ is a minimum

$$2(B_\lambda'-B_\lambda)-\sigma_\lambda=2\frac{r_\lambda}{\kappa_\lambda}\{2(1+\varpi_\lambda)-1\}=2\frac{r_\lambda}{\kappa_\lambda}(1+2\varpi_\lambda)$$

and therefore from (2) and (4) $C_{1,n}$ will be a minimum if

$$\Sigma\kappa_\lambda r_\lambda(1+2\varpi_\lambda)=\Sigma(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda\varpi_\lambda. \quad . \quad . \quad . \qquad (5)$$

It is convenient to satisfy this condition and to calculate the corresponding values of $C_{1,n}$ and $B'_{1,n}-B_{1,n}$. The calculation of the coefficients for any other conformation of the system may be derived immediately in terms of these quantities, of $\varpi_{1,n}$, and of the change in curvature which will transform the system from the one conformation to the other. When each component is an achromatic combination of the same two glasses the ϖ of every component and of the compound system has the value

$$\frac{K}{\mu}+\frac{K'}{\mu'},$$

where μ and μ' are the refractive indices of the crown and flint glasses. Since

$$\Sigma(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda=0, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

the condition (5) then takes the simple form

$$\Sigma\kappa_\lambda r_\lambda=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

This condition, together with the $n-1$ geometrical conditions already mentioned enables r_λ to be determined uniquely for all values of λ. Equations (2) and (3) may be simplified when the conditions $\varpi_\lambda=\varpi_{1,n}=\varpi$ are always true. In consequence of (6) equation (2) becomes

$$\kappa^2{}_{1,n}(B'_{1,n}-B_{1,n})=\Sigma\kappa_\lambda{}^2(B_\lambda'-B_\lambda). \quad . \quad . \quad . \qquad (8)$$

Again, $\kappa_{1,\lambda}-\kappa_{\lambda,n}=\kappa_{1,\lambda}+\kappa_{1,\lambda-1}-\kappa_{1,n}$

and $\Sigma\kappa_\lambda(\kappa_{1,\lambda}+\kappa_{1,\lambda-1})=\kappa^2_{1,n}$

and therefore

$$\begin{aligned}\Sigma\kappa_\lambda(\kappa_{1,\lambda}-\kappa_{\lambda,n})^2 & \\ &=\Sigma\kappa_\lambda(\kappa_{1,\lambda}-\kappa_{1,\lambda-1})^2+4\Sigma\kappa_\lambda\kappa_{1,\lambda}\kappa_{1,\lambda-1}-\kappa^3_{1,n}\\ &=\Sigma\kappa_\lambda{}^3+4\Sigma\kappa_\lambda\kappa_{1,\lambda}\kappa_{1,\lambda-1}-\kappa^3_{1,n}\end{aligned}$$

but

$$\begin{aligned}\Sigma\kappa^3&=\Sigma(\kappa_{1,\lambda}-\kappa_{1,\lambda-1})^3\\ &=\kappa^3_{1,n}-3\Sigma\kappa_\lambda\kappa_{1,\lambda}\kappa_{1,\lambda-1},\end{aligned}$$

and thus $3\Sigma\kappa_\lambda(\kappa_{1,\lambda}-\kappa_{\lambda,n})^2=\kappa^3_{1,n}-\Sigma\kappa_\lambda{}^3$. . . . . (9)

Substitute this value in (3). The result is

$$\begin{aligned}\kappa^3_{1,n}(3C_{1,n}+\varpi)&\\ &=\sum_1^n\{\kappa_\lambda{}^3(3C_\lambda+\varpi)-3(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda{}^2(B'_\lambda-B_\lambda)\} \quad . \quad . \quad (10)\end{aligned}$$

It is convenient to refer the components to a standard lens of the same "type." One thin lens may be said to be of the same type as another, when the same glasses are employed in the same order, and the first can be made geometrically similar to the second by the addition of the same curvature to each surface. The standard lens will be assumed to be of unit focal length and will have that form for which C is a minimum. The aberration coefficients of the standard lens will be distinguished by the suffix 0. The curvature of the first surface of the standard lens will be denoted by S, and that of the last surface by S'.

The $n-1$ geometrical conditions which must be satisfied when the components are all of the same type as the standard lens, the odd numbers the same way round as the standard and the even numbers reversed, are

$$\kappa_\lambda S'+r_\lambda=-\kappa_{\lambda+1}S'+r_{\lambda+1},$$

when λ is odd, and

$$-\kappa_\lambda S+r_\lambda=\kappa_{\lambda+1}S+r_{\lambda+1},$$

when λ is even.

These conditions, as well as (7) are satisfied if

$$\left.\begin{array}{l}2r_\lambda=-(\kappa_{1,\lambda}-\kappa_{\lambda+1,n})S+(\kappa_{1,\lambda-1}-\kappa_{\lambda,n})S'+\dfrac{\theta}{\kappa_{1,n}}(S+S')\\ \text{when } \lambda \text{ is odd, and}\\ 2r_\lambda=-(\kappa_{1,\lambda-1}-\kappa_{\lambda,n})S+(\kappa_{1,\lambda}-\kappa_{\lambda+1,n})S'+\dfrac{\theta}{\kappa_{1,n}}(S+S')\\ \text{when } \lambda \text{ is even, where}\end{array}\right\}.(11)$$

$$\theta=\kappa_1{}^2-\kappa_2{}^2+\kappa_3{}^2-\kappa_4{}^2+\ .\ .\ .\ .\ .\ .\ . \quad (12)$$

The aberration coefficients of the components are connected with those of the standard lens by the two equations

$$\left.\begin{array}{ll} & C_\lambda=C_0+\left(\frac{r_\lambda}{\kappa_\lambda}\right)^2(1+2\varpi) \\ \text{and} & B'_\lambda-B_\lambda=B_0'-B_0+2\frac{r_\lambda}{\kappa_\lambda}(1+\varpi) \\ \text{when } \lambda \text{ is odd. and} & \\ & B'_\lambda-B_\lambda=-(B_0'-B_0)+2\frac{r_\lambda}{\kappa_\lambda}(1+\varpi) \\ \text{when } \lambda \text{ is even.} & \end{array}\right\} . \quad (13)$$

Use large heavy type for the aberration coefficients of the complete lens, and let the suffix 0 be added when equation (7) is satisfied. Substitution from (13) in equation (8) gives

$$\kappa^2_{1,n}(\mathbf{B}_0'-\mathbf{B}_0)=\theta(B_0'-B_0) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Let φ be written for

$$\Sigma(-)^\lambda(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa^2_\lambda. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Then

$$\Sigma(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda^2(B'_\lambda-B_\lambda)$$
$$=-\varphi(B_0'-B_0)+2(1+\varpi)\Sigma(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda r_\lambda,$$

and

$$\Sigma\kappa_\lambda^3(3C_\lambda+\varpi_\lambda)=(3C_0+\varpi)\Sigma\kappa_\lambda^3+3(1+2\varpi)\Sigma\kappa_\lambda r_\lambda^2.$$

Now

$$2\Sigma(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda r_\lambda$$
$$=(S'-S)\Sigma(\kappa_{1,\lambda}-\kappa_{\lambda,n})^2\kappa_\lambda+(S+S')\Sigma(-)^\lambda(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda^2$$
$$+(S+S')\frac{\theta}{\kappa_{1,n}}\Sigma(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda$$
$$=\tfrac{1}{3}(S'-S)(\kappa^3_{1,n}-\Sigma\kappa_\lambda^3)+\varphi(S+S')$$

by (6), (9) and (15).

Also

$$4\Sigma\kappa_\lambda r_\lambda^2$$
$$=(S'-S)^2\Sigma(\kappa_{1,\lambda}-\kappa_{\lambda,n})^2\kappa_\lambda+(S+S')^2\Sigma\kappa_\lambda^3+(S+S')^2\frac{\theta^2}{\kappa^2_{1,n}}\Sigma\kappa_\lambda$$
$$+2(S'^2-S^2)\Sigma(-)^\lambda(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda^2-2(S+S')^2\frac{\theta^2}{\kappa_{1,n}}$$
$$+2(S'^2-S^2)\frac{\theta}{\kappa_{1,n}}\Sigma(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda$$
$$=\tfrac{1}{3}(S'-S)^2(\kappa^3_{1,n}-\Sigma\kappa_\lambda^3)+(S+S')^2\left(\Sigma\kappa_\lambda^3-\frac{\theta^2}{\kappa_{1,n}}\right)$$
$$+2\varphi(S'^2-S^2),$$

and, therefore, from (10)

$$\begin{aligned}
&4\kappa^3_{1,n}(3\mathbf{C}_0+\varpi)\\
&=4(3C_0+\varpi)\Sigma\kappa_\lambda{}^3+12\varphi(B_0'-B_0)\\
&\quad+(1+2\varpi)\Big\{(S'-S)^2(\kappa^3_{1,n}-\Sigma\kappa_\lambda{}^3)+3(S+S')^2\Big(\Sigma\kappa_\lambda{}^3-\frac{\theta^2}{\kappa_{1,n}}\Big)\\
&\qquad\qquad+6\varphi(S'^2-S^2)\Big\}\\
&\quad-4(1+\varpi)\{(S'-S)(\kappa^3_{1,n}-\Sigma\kappa_\lambda{}^3)+3\varphi(S+S')\}.
\end{aligned}$$

Let A denote the spherical aberration coefficient for a component having its first surface plane, and similarly A' the reversed spherical aberration coefficient for a component having its last surface plane. These coefficients may be found by impressing the general curvatures $-S$ and $-S'$ on the standard lens. Therefore,

$$\begin{aligned}
A&=A_0+2S(1+\varpi)+S^2(1+2\varpi)\\
&=-(B_0'-B_0)+C_0+1+\varpi+2S(1+\varpi)+S^2(1+2\varpi),
\end{aligned}$$

and

$$\begin{aligned}
A'&=A_0'-2S'(1+\varpi)+S'^2(1+2\varpi)\\
&=B_0'-B_0+C_0+1+\varpi-2S'(1+\varpi)+S'^2(1+2\varpi),
\end{aligned}$$

from which it may be seen that the coefficient of φ in the above expression is $6(A'-A)$.

The coefficient of $\kappa^3_{1,n}-\Sigma\kappa_\lambda{}^3$ is

$$\begin{aligned}
&(1+2\varpi)(S'-S)^2-4(1+\varpi)(S'-S)\\
&=-(1+2\varpi)(S'+S)^2+2\{(1+2\varpi)(S'^2+S^2)-2(1+\varpi)(S'-S)\}\\
&=-(1+2\varpi)(S'+S)^2+2\{A+A'-2C_0-2\varpi-2\},
\end{aligned}$$

and the equation may therefore be written

$$\begin{aligned}
4\kappa^3_{1,n}(3\mathbf{C}_0+\varpi)=&\,4(3C_0+\varpi)\Sigma\kappa_\lambda{}^3\\
&+2(A+A'-2C_0-2\varpi-2)(\kappa^3_{1,n}-\Sigma\kappa_\lambda{}^3)\\
&+(S+S')^2(1+2\varpi)\Big(4\Sigma\kappa_\lambda{}^3-\kappa^3_{1,n}-\frac{3\theta^2}{\kappa_{1,n}}\Big)\\
&+6(A'-A)\varphi,
\end{aligned}$$

or, somewhat more symmetrically

$$\begin{aligned}
&\{4\mathbf{C}_0+2\varpi+1-\tfrac{1}{2}(A+A')\}\kappa^3_{1,n}+(S+S')^2(1+2\varpi)\frac{\theta^2}{\kappa_{1,n}}\\
&=\tfrac{1}{3}\{4C_0+2\varpi+1-\tfrac{1}{2}(A+A')+(S+S')^2(1+2\varpi)\}(4\Sigma\kappa_\lambda{}^3-\kappa^3_{1,n})\\
&\qquad\qquad+2(A'-A)\varphi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)
\end{aligned}$$

Equations (14) and (16) express the aberration coefficients of the compound lens in terms of those of the standard lens. The curvatures of the external surfaces of the compound lens are $\kappa_1 S+r_1$ and $\kappa_1 S'+r_n$ if n is odd, or $-\kappa_n S+r_n$ if n is even, *i.e.*, the curvature of the first surface is

$$\tfrac{1}{2}\left\{\kappa_{1,n}(S-S')+\frac{\theta}{\kappa_{1,n}}(S+S')\right\}; \quad . \; . \; . \; . \; . \; . \; . \quad (17)$$

and that of the last surface is

$$\tfrac{1}{2}\left\{-\kappa_{1,n}(S-S')+\frac{\theta}{\kappa_{1,n}}(S+S')\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (18)$$

The curvatures of intermediate surfaces are most simply obtained from these by noting that the successive surfaces differ from one another in curvature by the amounts

$$\kappa_1 R,\ (\kappa_1+\kappa_2)R',\ (\kappa_2+\kappa_3)R,\ (\kappa_3+\kappa_4)R'. \quad . \; . \; . \; . \; . \quad (19)$$

Application to the Calculation of Objectives.

The two aberration conditions which an achromatic thin lens can satisfy may always be expressed in the form that $\mathbf{C}$ and $\mathbf{B}'-\mathbf{B}$ are to have definite values. The most frequent conditions are the absence of first order spherical aberration and coma for a definite magnification. Denote this magnification by m. The conditions to be satisfied are

$$4\mathbf{C}+2\varpi+1=\left(\frac{1+m}{1-m}\right)^2(5+2\varpi),$$

and

$$\mathbf{B}'-\mathbf{B}=\frac{1+m}{1-m}(2+\varpi).$$

These values are to be derived by bending the lens from the form in which $\mathbf{C}$ is a minimum, and require the relation

$$\left\{\frac{1+m}{1-m}+(\mathbf{B}'_0-\mathbf{B}_0)(1+2\varpi)(2+\varpi)\right\}^2$$
$$=(1+\varpi)^2\{(\mathbf{B}'_0-\mathbf{B}_0)^2(5+2\varpi)(1+2\varpi)+4\mathbf{C}_0+2\varpi+1\}$$

between the aberration coefficients of the unbent lens to be satisfied. This condition implies that a relation must be satisfied by the κ's, and this relation is evidently of the fourth order. It will be shown that it can always be reduced to a linear condition connecting two κ's, and that the determination of a lens satisfying the two aberration conditions requires the solution of a quadratic equation. This might be inferred from the case of triple objectives, since a triple objective may be made to add any required amount to the aberrations arising in the rest of the system.

The simplest method of solution is to assume definite values for $\kappa_{1,n}$ and for θ. The curvature r must then be added to each of the curvatures given by equations (17), (18) and (19), where

$$B'-B=B'_0-B_0+2\frac{r}{\kappa_{1,n}}(1+\varpi).$$

It is evident from this equation that all lenses satisfying the same conditions have their r and θ connected by a linear equation. This is equivalent to saying that the r and θ are determined by the curvatures of the external surfaces of the system, and these may, if preferred, be taken as the independent variables. To complete the solution it is necessary to choose the κ's so that the value of C_0 given by equation (16) agrees with that which makes

$$C=C_0+\left(\frac{r}{\kappa_{1,n}}\right)^2(1+2\varpi).$$

It is obvious from the equations which have been found that if θ, r are values which satisfy the conditions when a crown component leads, the corresponding values with a flint component leading, and unaltered curvatures for the first and last surfaces, are $-\theta$, r.

Quadruple Objectives.

Take first a system consisting of two crown and two flint lenses. The identities

$$\theta=\kappa_1^2-\kappa_2^2+\kappa_3^2=\kappa^2_{1,3}-2(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)$$

$$\kappa_1^3+\kappa_2^3+\kappa_3^3=\kappa^3_{1,3}-3(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)(\kappa_3+\kappa_1)$$

and

$$\begin{aligned}\varphi&=\kappa_1^2(\kappa_2+\kappa_3)+\kappa_2^2(\kappa_1-\kappa_3)-\kappa_3^2(\kappa_1+\kappa_2)\\&=(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)(\kappa_1-\kappa_3),\end{aligned}$$

show that if definite values are assigned to $\kappa_{1,3}$ and θ, equation (16) becomes a linear relation between κ_1 and κ_3. From this relation and from the assumed value of $\kappa_{1,3}$ the powers of two of the component doublets may be expressed as linear functions of the power of the remaining component. There are therefore two solutions when $\kappa_{1,3}$ and θ are given. There will be two solutions to the corresponding problem when a flint lens leads instead of a crown. With two given glasses there are therefore four infinite series of objectives possible.

The four triple objectives will occur as the special solutions $\kappa_1=0$ and $\kappa_3=0$ in both crown leading and flint leading solutions. $\kappa_2=0$ is, of course, not a triple objective.

The linear relation between κ_1 and κ_3 in this case may be written

$$\{4C_0+2\varpi+1+(S+S')^2(1+2\varpi)-A'\}\kappa_1 \\ +\{4C_0+2\varpi+1+(S+S')^2(1+2\varpi)-A\}\kappa_3 \\ =\frac{(C_0-\mathbf{C}_0)\kappa^3_{1,3}}{(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)}+(S+S')^2(1+2\varpi)\left\{\kappa_{1,3}-\frac{(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)}{\kappa_{1,3}}\right\}$$

It is evident that $(S+S')^2$ only disappears from the equation if κ_1 or κ_3 or $\kappa_1+\kappa_2$ or $\kappa_2+\kappa_3$ vanishes. In the first case the equation becomes

$$(4\mathbf{C}_0+2\varpi+1-A)\kappa^2_{2,3}=(4C_0+2\varpi+1-A)(\kappa_2-\kappa_3)^2,$$

and in the second

$$(4\mathbf{C}_0+2\varpi+1-A')\kappa^2_{1,2}=(4C_0+2\varpi+1-A')(\kappa_1-\kappa_2)^2,$$

the equations previously found for triple objectives.

In the last two cases the system reduces to a doublet and the equation becomes $\mathbf{C}_0=C_0$.

Quintuple Objectives.

The solution of quintuple objectives may be carried out in a very similar way to that used in the previous case. If $\kappa_3+\kappa_4=0$, $\kappa_{1,4}$, θ, and $\Sigma\kappa_\lambda{}^3$ reduce to $\kappa_1+\kappa_2$, $\kappa_1^2-\kappa_2^2$, and $\kappa_1^3+\kappa_2^3$ respectively, and therefore

$$\kappa^4_{1,4}+3\theta^2-4\kappa_{1,4}\Sigma\kappa_\lambda{}^3$$

vanishes. Thus $\kappa_3+\kappa_4$ is a factor of this expression, and similarly $\kappa_1+\kappa_2$, $\kappa_2+\kappa_3$, and $\kappa_1+\kappa_4$ must be factors. When $\kappa_1=\kappa_2=\kappa_3=\kappa_4=1$, the value of the expression is 3×4^3, and therefore

$$\kappa^4_{1,4}+3\theta^2-4\kappa_{1,4}\Sigma\kappa_\lambda{}^3=12(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)(\kappa_3+\kappa_4)(\kappa_1+\kappa_4).$$

Again for a triplet $\varphi=\kappa_1\kappa_2(\kappa_1+\kappa_2)$, and the present system reduces to a triplet if $\kappa_1+\kappa_2$, $\kappa_2+\kappa_3$, or $\kappa_3+\kappa_4$ vanishes. Therefore these are all factors of

$$\kappa^3_{1,4}-\Sigma\kappa_\lambda{}^3-3\varphi.$$

When each κ is equal to unity the value of this expression is 3×4^2, and thus

$$\kappa^3_{1,4}-\Sigma\kappa_\lambda{}^3-3\varphi=6(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)(\kappa_3+\kappa_4).$$

If then definite values are assigned to $\kappa_{1,4}$, θ and either $\Sigma\kappa_\lambda{}^3$ or φ, the condition (16) becomes a simple equation for $\kappa_1+\kappa_4$.

From the values of $\kappa_1+\kappa_4$, $\kappa_{1,4}$, and θ, any three of the four quantities κ_1, κ_2, κ_3, κ_4 are found as linear functions of the fourth, and either of the two identities given above becomes a quadratic equation for the determination of this κ. There are thus the same number of series of objectives as before, but each series is now doubly infinite.

The equation for $\kappa_1+\kappa_4$ when $\kappa_{1,4}$, θ and $(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)(\kappa_3+\kappa_4)$ are taken as independent variables is

$$\begin{aligned}&(4C_0+2\varpi+1-A')\kappa^4{}_{1,4}-(4C_0+2\varpi+1-A')\theta^2\\&\quad+4(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)(\kappa_3+\kappa_4)[\{4C_0+2\varpi+1\\&\qquad+(S+S')^2(1+2\varpi)-A'\}(\kappa_1+\kappa_4)+\overline{(A'-A)}\kappa_{1,4}]=0.\end{aligned}$$

Multiple Objectives.

The results obtained in the case of objectives of four and five lenses may be shown to hold generally. Put

$$\begin{aligned}\psi=&(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)(\kappa_3+\kappa_4+\;.\;.\;.\;.\;.\;.\;.\;+\kappa_n)\\&+(\kappa_1+\kappa_2+\kappa_3+\kappa_4)(\kappa_4+\kappa_5)(\kappa_5+\kappa_6+\;.\;.\;.\;.\;.\;+\kappa_n)\\&+(\kappa_1+\kappa_2+\;.\;.\;+\kappa_5+\kappa_6)(\kappa_6+\kappa_7)(\kappa_7+\kappa_8+\;.\;.\;.\;+\kappa_n)\\&+\;.\;.\;.\;.\end{aligned}$$

and

$$\begin{aligned}\psi'=&\kappa_1(\kappa_1+\kappa_2)(\kappa_2+\kappa_3+\kappa_4+\;.\;.\;.\;.\;.\;.\;.\;+\kappa_n)\\&+(\kappa_1+\kappa_2+\kappa_3)(\kappa_3+\kappa_4)(\kappa_4+\kappa_5+\;.\;.\;.\;.\;.\;+\kappa_n)\\&+(\kappa_1+\kappa_2+\kappa_3+\kappa_4+\kappa_5)(\kappa_5+\kappa_6)(\kappa_6+\kappa_7+\;.\;.\;.\;+\kappa_n)\\&+\;.\;.\;.\;,\end{aligned}$$

the general term in each case being

$$\kappa_{1,\lambda}(\kappa_\lambda+\kappa_{\lambda+1})\kappa_{\lambda+1,n}$$

where in ψ, λ is given all even values less than n, and in ψ' all odd values below the same limit. Then

$$\begin{aligned}\psi+\psi'&=\sum_1^n\kappa_\lambda(\kappa_{1,\lambda}\kappa_{\lambda+1,n}+\kappa_{1,\lambda-1}\kappa_{\lambda,n})\\&=\kappa_{1,n}\Sigma\kappa_\lambda(\kappa_{1,\lambda}+\kappa_{1,\lambda-1})-\Sigma\kappa_\lambda(\kappa^2{}_{1,\lambda}+\kappa^2{}_{1,\lambda-1})\\&=\kappa^3{}_{1,n}-\Sigma\kappa_\lambda{}^3-2\Sigma\kappa_\lambda\kappa_{1,\lambda}\kappa_{1,\lambda-1}\\&=\tfrac{1}{3}(\kappa^3{}_{1,n}-\Sigma\kappa_\lambda{}^3)\end{aligned}$$

by a result already proved, and

$$\begin{aligned}\psi'-\psi&=-\Sigma(-)^{\lambda}\kappa_{\lambda}\{\kappa_{1,\lambda}\kappa_{\lambda+1,n}-\kappa_{1,\lambda-1}\kappa_{\lambda,n}\}\\&=-\Sigma(-)^{\lambda}\kappa_{\lambda}\{\kappa_{1,\lambda}(\kappa_{\lambda,n}-\kappa_{\lambda})-(\kappa_{1,\lambda}-\kappa_{\lambda})\kappa_{\lambda,n}\}\\&=\Sigma(-)^{\lambda}\kappa_{\lambda}^{2}(\kappa_{1,\lambda}-\kappa_{\lambda,n})\\&=\varphi.\end{aligned}$$

Substitute for φ and $\Sigma\kappa_{\lambda}^{3}$ in (16) from these expresisons. This equation may then be written,

$$\begin{aligned}&4(\mathbf{C}_0-C_0)\kappa^{4}_{1,n}+(S+S')^2(1+2\varpi)(\theta^2-\kappa^{4}_{1,n})\\&+4(4C_0+2\varpi+1+(S+S')^2(1+2\varpi)-A)\psi\kappa_{1,n}\\&+4(4C_0+2\varpi+1+(S+S')^2(1+2\varpi)-A')\psi'\kappa_{1,n}\\&\qquad=0.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(20)\end{aligned}$$

Now, assume that $\kappa_{1,n}$, θ, and all but three consecutive κ's have been fixed arbitrarily, or what comes to the same thing, that the curvatures of all but four successive surfaces are given. Let the unknown κ's be $\kappa_{\lambda-1}$, κ_{λ}, and $\kappa_{\lambda+1}$. From $\kappa_{1,n}$ the value of $\kappa_{\lambda-1}+\kappa_{\lambda}+\kappa_{\lambda+1}$ is known, and from θ that of $\kappa^{2}_{\lambda-1}-\kappa^{2}_{\lambda}+\kappa^{2}_{\lambda+1}$ is known. Since the last is equal to

$$(\kappa_{\lambda-1}+\kappa_{\lambda}+\kappa_{\lambda+1})^2-2(\kappa_{\lambda-1}+\kappa_{\lambda})(\kappa_{\lambda}+\kappa_{\lambda+1})$$

it follows that $(\kappa_{\lambda-1}+\kappa_{\lambda})(\kappa_{\lambda}+\kappa_{\lambda+1})$ is known. The unknown contributions to ψ and ψ' are in the one case

$$\begin{aligned}&(\kappa_1+\kappa_2+\ldots+\kappa_{\lambda-2})(\kappa_{\lambda-2}+\kappa_{\lambda-1})(\kappa_{\lambda-1}+\kappa^{\lambda}+\quad.\quad.\quad.\quad+\kappa_n)\\&+(\kappa_1+\kappa_2+\ldots+\kappa_{\lambda})(\kappa_{\lambda}+\kappa_{\lambda+1})(\kappa_{\lambda+1}+\kappa_{\lambda+2}+\quad.\quad.\quad.\quad+\kappa_n),\end{aligned}$$

and in the other

$$\begin{aligned}&(\kappa_1+\kappa_2+\ldots+\kappa_{\lambda-1})(\kappa_{\lambda-1}+\kappa_{\lambda})(\kappa_{\lambda}+\kappa_{\lambda+1}+\quad.\quad.\quad.\quad.\quad+\kappa_n)\\&+(\kappa_1+\kappa_2+\ldots+\kappa_{\lambda+1})(\kappa_{\lambda+1}+\kappa_{\lambda+2})(\kappa_{\lambda+2}+\kappa_{\lambda+3}+\quad.\quad.\quad+\kappa_n).\end{aligned}$$

In the first and last of these four lines the first and last factors are known in each case, so that the first line is of the form $a\kappa_{\lambda-1}+b$, and the last of the form $c\kappa_{\lambda+1}+d$, where a, b, c, d are known. Since $(\kappa_{\lambda-1}+\kappa_{\lambda})(\kappa_{\lambda}+\kappa_{\lambda+1})$ is known the second line may be written

$$e+f(\kappa_{\lambda}+\kappa_{\lambda+1})+g\kappa_{\lambda+1}+h(\kappa_{\lambda}+\kappa_{\lambda+1})\kappa_{\lambda+1},$$

or, since $\kappa_{\lambda-1}+\kappa_{\lambda}+\kappa_{\lambda+1}$ is known, it takes the form

$$e'+f'\kappa_{\lambda-1}+g\kappa_{\lambda+1}+h(\kappa_{\lambda}+\kappa_{\lambda+1})\kappa_{\lambda+1}.$$

Now

$$\begin{aligned}\kappa^{2}_{\lambda-1}-\kappa_{\lambda}^{2}+\kappa^{2}_{\lambda+1}+(\kappa_{\lambda-1}+\kappa_{\lambda}+\kappa_{\lambda+1})^2\\=2\kappa_{\lambda-1}(\kappa_{\lambda-1}+\kappa_{\lambda}+\kappa_{\lambda+1})+2(\kappa_{\lambda}+\kappa_{\lambda+1})\kappa_{\lambda+1},\end{aligned}$$

and the second line is thus of the form

$$e''+f''\kappa_{\lambda-1}+g''\kappa_{\lambda+1}.$$

The third line may be shown to be of a similar form, and, therefore, both ψ and ψ' are of the form

$$E + F\kappa_{\lambda-1} + G\kappa_{\lambda+1},$$

that is to say, equation (16) or (20) is a linear relation between $\kappa_{\lambda-1}$ and $\kappa_{\lambda+1}$. It follows that the solution is completed by finding the roots of a quadratic equation as in previous cases.

The formula (20) is easily remembered if it is noted that ψ multiplies the term containing A, the spherical aberration of a doublet with a flat crown surface leading, and consists of a series of terms each of which is the sum of the κ's which determine the power of a crown lens in the compound system, multiplied by the sum of all κ's relating to preceding lenses, and by the sum of all κ's relating to succeeding lenses. The other term is determined in a similar way by substituting "flint" for "crown."

Calculation of the Fundamental Constants.

It will usually happen that the glasses from which multiple objectives are made will be restricted to a very small number of varieties, the same two glasses being frequently used for many different purposes. The foregoing analysis shows that very few numbers are required in the calculation of any multiple objectives from a cemented doublet, and it is convenient to keep these in a form suitable for immediate reference. The author has found it best to enter these quantities on a card of the stock size used in card-filing cabinets. The amount of space on these is very limited, and it is consequently best to pay some regard to the length of the expressions to be tabulated in deciding upon the most convenient arrangement of the card. The order adopted is in consequence slightly different from the natural order in which the values are obtained.

The arrangement adopted is shown below. The top line defines the glasses to which the card relates, and also contains a reference to the detailed calculations. It is sometimes more convenient to enter log ν/ν' rather than ν/ν', and space has been left for the insertion of log in such cases. The two quantities first determined are K and K', and these are entered in the first column. The next quantities R and R' are most conveniently placed at the bottom of the second column. The absolute first order coefficients for the doublets occupy the second line, and the curvatures of this standard lens complete the first column. The remaining space in the second column

$\mu =$	$\mu' =$	$\nu/\nu' =$	*Ref.*
$K =$	$\varpi =$	$4C_0+2\varpi+1=$	$B'_0-B_0=$
$K'=$	$A =$	$4C_0+2\varpi+1-A =$	$\frac{B'_0-B_0}{2(1+\varpi)}=$
$R_1=$	$A' =$	$4C_0+2\varpi+1-A'=$	$\frac{2+\varpi}{2(1+\varpi)}=$
$R_2=$	$R =$	$(R_1+R_3)^2(1+2\varpi)=$	$5+2\varpi=$ $4(1+2\varpi)=$
$R_3=$	$R'=$	$\frac{A-A'-B+B'}{2(1+\varpi)}=$	$\frac{R_1-R_3}{2}=$
$4C_0+2\varpi+1+(R_1+R_3)^2(1+2\varpi)-A=$		$4C_0+2\varpi+1+(R_1+R_3)^2(1+2\varpi)-A'=$	

is taken up by A and A', which may be calculated from the formulæ given below, and checked by derivation from the coefficients for the standard lens. These are conveniently placed to enable the quantities required in the calculation of triple objectives to be written down in the middle of the third column. The additional term required for more complex objectives is entered under this, and the two long expressions obtained by adding this to the terms above are entered below the regular columns. The quantities in the last column assist in the determination of the curvatures and in finding the correct value for $\mathbf{C}_0$. There is further space left on the card for the entry of any other numbers which may be frequently required in connection with the two glasses concerned. R_1 and R_3 have been substituted for S and S'.

A may be calculated from any of the formulæ

$$A=(R+R'+1)^2+\frac{\mu'-\mu}{\mu'^2}R\{(R+R'+1)^2-(R+K)^2\}$$

$$=(R+R'+1)^2+\frac{\mu'-\mu}{\mu'}RR'\{2R+R'+1+K\}$$

$$=(R+R'+1)^2(2\frac{K'}{\mu'}+1)-R'\{(1+K')(R+R'+1)+(K+R)\}$$

with similar expressions for A'. The results may be checked from the values of C_0 and B'_0-B_0, or from one of the relations

$$\begin{aligned}4C_0+2\varpi+1+(R_1+R_3)^2(1+2\varpi)&\\ =A+A'-(R+R'+1)^2+(R-R')(K'R-KR')&\\ =A+A'-(2R+1)(2R'+1)-(R-R')(KR-K'R')&\\ =2(A+A')-(R+R'+1)^2(1+2\varpi)-2(R+R'+1).&\end{aligned}$$

Chromatic Differences of First Order Aberrations.

Equations (14) and (20) are expressions for the values of $\mathbf{B}'_0-\mathbf{B}_0$ and $\mathbf{C}_0$, the fundamental aberration coefficients of the thin compound lens in terms of B'_0-B_0 and C_0, the corresponding coefficients of the standard component. The quantities which occur in these equations, $\kappa_{1..n}$, θ, ψ, and ψ' are simple ratios, and depend only on the way in which the compound lens is built up from its components. They have no connection with the aberrations of the components or with the wave length which is being considered. When a compound lens has been calculated to satisfy given conditions for light of a definite wave-length, the values of $\mathbf{B}'_0-\mathbf{B}_0$ and $\mathbf{C}_0$ for the same compound lens, but for another wave-length are found, using the same values for $\kappa_{1,n}$, θ, ψ and ψ', and new values for B'_0-B_0, C_0, ϖ, S and S'. The value of r to be used in calculating $\mathbf{B}'-\mathbf{B}$ and $\mathbf{C}$ from $\mathbf{B}'_0-\mathbf{B}_0$ and $\mathbf{C}_0$ will, as a rule, vary with the colour. If the power of the component for the new colour is k when it is unity for the first colour, and the quantities relating to the new colour are distinguished by a bar above the letter, equations (17) and (18) show that

$$k\left\{\bar{r}+\frac{\theta}{2\kappa_{1,n}}(\bar{S}+\bar{S}')\right\}=r+\frac{\theta}{2\kappa_{1,n}}(S+S'),$$

and

$$k(\bar{S}-\bar{S}')=S-S'.$$

The aberration coefficients for the new colour are given by

$$\bar{\mathbf{B}}'-\bar{\mathbf{B}}=\bar{\mathbf{B}}'_0-\bar{\mathbf{B}}_0+2\frac{\bar{r}}{\kappa_{1,n}}(1+\bar{\varpi}),$$

and

$$\bar{\mathbf{C}}=\bar{\mathbf{C}}_0+\left(\frac{\bar{r}}{\kappa_{1,n}}\right)^2(1+2\bar{\varpi}).$$

Evidently all lenses having the same values for the ratios $\kappa^3_{1,n}:\theta\kappa_{1,n}:\psi:\psi'$ will have the same chromatic differences of first order aberrations if made from the same kinds of glass.

Illustrative Quadruple Lenses.

By way of examples a number of quadruple lenses have been calculated. The ratio taken for the power of the crown glass to that of the flint is $10^{0\cdot 20}$ and the refractive index of the flint for the standard line is 1·62. In one series the refractive index for the crown is 1·50, and in another series 1·55. Fig. 1 illustrates the composition of the lenses for various values of r when the former crown glass is taken and a crown lens leads. The most interesting branches of the curves, those which contain the triple objectives, are closed. On one side there are infinite branches to the curves, but as the systems to which these correspond involve the use of more powerful elements than a cemented doublet of equal focal length they are not likely to be adopted in practice. The closed curves and the infinite branches are separated by a belt limited by two values of r between which there is no real solution. For values of r greater than the extreme point on the closed curve there is no real solution. It is obvious that any branch of κ_1 or of κ_3 must lie wholly on one side of $\kappa=1$, since this value of κ corresponds to a doublet. As the refractive indices are changed towards a combination in which a doublet satisfies the first order conditions, the curves for κ_1 and κ_3 will become more pointed, and when the doublet form is reached it appears probable that the closed and infinite branches of the curve concerned will meet in a node on the line $\kappa=1$. A slight alteration in one of the glasses in either direction will suffice for the connection between the branches to be broken.

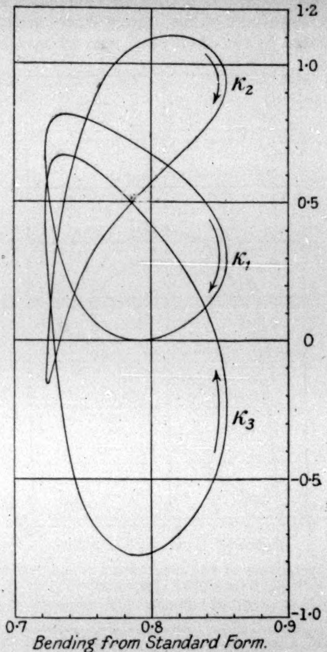

FIG. 1.—COMPOSITION OF QUADRUPLE OBJECTIVES WITH CROWN LENS LEADING IN TERMS OF DOUBLETS.

Refractive indices 1·50 and 1·62. Log $\nu/\nu'=0{\cdot}20$.

It will be noted that while κ_3 crosses the line $\kappa=0$ at widely separated points, κ_1 is almost tangential to this line. This particular combination of glasses is thus near the limit at which two of the triple solutions become imaginary.

Fig. 2 shows the curvatures of the five surfaces of the quadruple systems. It has already been noted that the external curvatures are linear functions of r, and are thus represented by double straight lines. In the region where the κ's form closed curves, the curvatures of the inner surfaces are also of necessity represented by closed curves. Arrows have been inserted in Figs. 1 and 2 to denote the side of the closed curves which belong to the same series of objectives. The triple objectives in Fig. 2 correspond to the points where R_1 meets R_2 and R_5 meets R_4. Apart from the triple objectives there are 28 objectives which require for their production less than five different finite curvatures. The curves for R_2, R_3 and R_4 all cross the zero line, giving six objectives with flat surfaces. R_5 only just fails to reach this line, but can be made to cross it by a

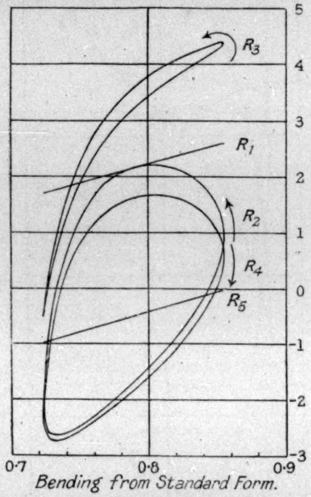

FIG. 2.—CURVATURES OF THE SURFACES OF QUADRUPLE OBJECTIVES WITH CROWN LENS LEADING.

Refractive indices 1·50 and 1·62. Log $\nu/\nu' = 0·20$.

change in the glasses employed. There are also two points of intersection of R_1 and R_3, two of R_2 and R_4, and two of R_2 and R_5. For somewhat different glasses there will be further intersections between R_1 and R_4, and R_3 and R_5. If the diagram is folded about the line $R=0$ further intersections, corresponding to curvatures of equal amounts but opposite signs, will be obtained. Inspection of the figure shows that R_1 and R_5 will not intersect, and that no further intersections between R_1 and R_3 will be obtained. With these exceptions there will be two new intersections between every pair of curves, giving 16 objectives having two curvatures of equal amount, but of opposite sign. These forms are to be preferred if they are not

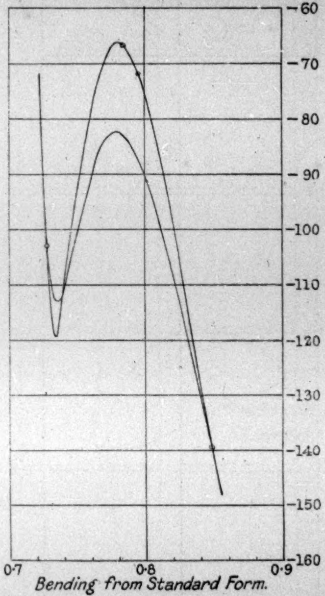

FIG. 3.—SECOND ORDER SPHERICAL ABERRATION FOR OBJECTIVES WITH CROWN LENS LEADING.

appreciably worse than others as regards residual aberrations. A few particularly simple systems are almost attained by the

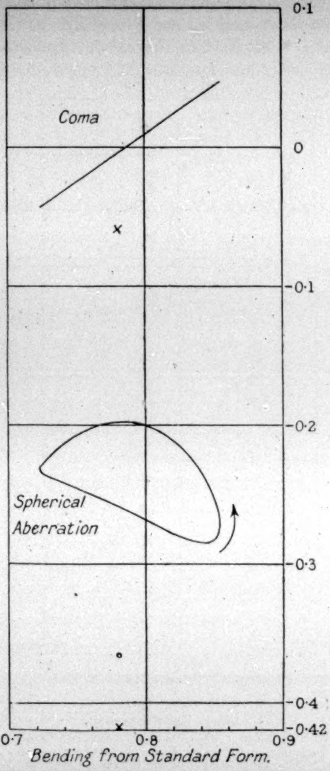

FIG. 4.—CHROMATIC DIFFERENCES OF FIRST ORDER SPHERICAL ABERRATION AND COMA FOR QUADRUPLE OBJECTIVES WITH CROWN LENS LEADING.

use of these glasses. For instance, $R_3=0$ in combination with $R_2=R_5$; also $R_1=R_3$ with $R_4=-R_5$. A particularly favourable form with different glasses would be $R_2 = R_4$ and $R_3 = R_5$. This is the case requiring minimum curvature on the steepest surfaces ; the two flint lenses are exactly alike, and the two crown lenses have one curvature alike. It is also one of the most favourable forms as regards second order spherical aberration.

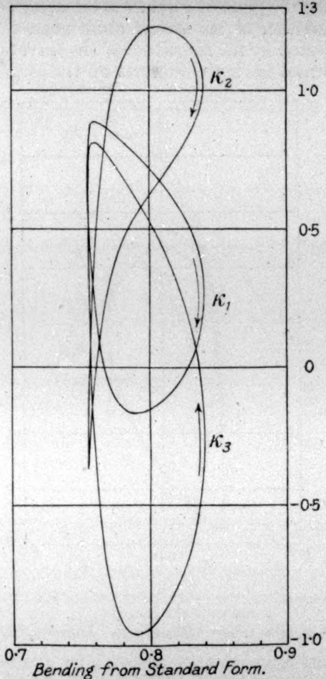

FIG. 5.—COMPOSITION OF QUADRUPLE OBJECTIVES WITH CROWN LENS LEADING IN TERMS OF DOUBLETS.
Refractive indices 1·55 and 1·62. Log $\nu/\nu' = 0{\cdot}20$.

Figure 3 shows the second order spherical aberration for the lenses to which Figures 1 and 2 relate. The triple objective points are distinguished by a circle. It will be observed that the two branches cross one another three times, and that one of the triple objectives is very nearly as good as the best corrected system. All the objectives are over-corrected. Those having the greatest curvatures are decidedly the worst, but the variation of the second order aberration is by no means indicated by the magnitude of the curvatures, the best corrected lenses having curvatures on the large side of the mean. Apart from its want of simplicity the most notable character of this curve is the rapid improvement in the neighbourhood of the minimum curvature, where the curves for R_2 and R_4 cross one another.

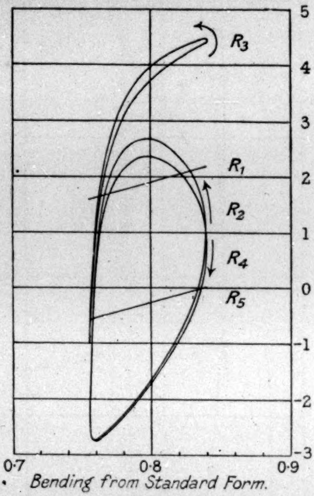

FIG. 6.—CURVATURES OF THE SURFACES OF QUADRUPLE OBJECTIVES WITH CROWN LENS LEADING.
Refractive indices 1·55 and 1·62. Log $\nu/\nu' = 0·20$.

Figure 4 gives the amounts of first order spherical aberration and coma for light of a different colour. In calculating these curves the refractive indices are taken as 1·50835 and 1·63641, these figures giving the same paraxial focus as 1·50 and 1·62.

The curve for the coma, which depends only on the external curvatures, is a double straight line. The arrow shows the direction in which the spherical aberration curve is described.

Figures 5 and 6 show how 1 and 2 are changed when the refractive index of the crown glass is increased to 1·55, corresponding approximately to the substitution of a medium barium crown glass for a borosilicate crown. The closed curves extend over a much smaller range of values of r, but

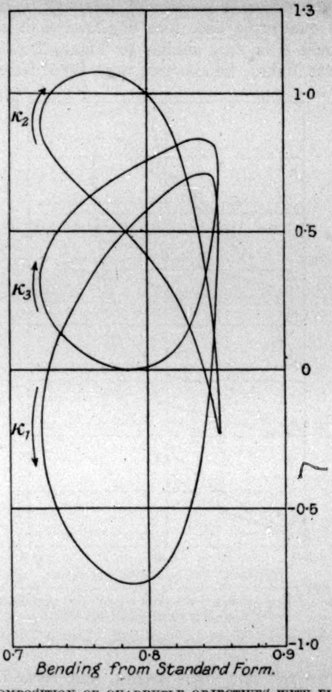

FIG. 7.—COMPOSITION OF QUADRUPLE OBJECTIVES WITH FLINT LENS LEADING IN TERMS OF DOUBLETS.

Refractive indices 1·50 and 1·62. Log $\nu/\nu' = 0·20$.

the curvatures of the various surfaces lie within almost unchanged limits.

All the foregoing diagrams relate to quadruple objectives with a crown component leading. Figures 7, 8, 9 and 10 correspond to 1, 2, 3 and 4, but with a flint lens in front. The two sets of curves necessarily intersect in the positions where the quadruple objective degenerates into the triple form. Figure 8 is very similar to Figure 2 as regards the closed curves if these be assumed to be lifted from the paper, revolved through two right angles, and replaced with the double straight lines on the same lines as before. It follows that the least curvatures are to be found in the form with a crown lens leading.

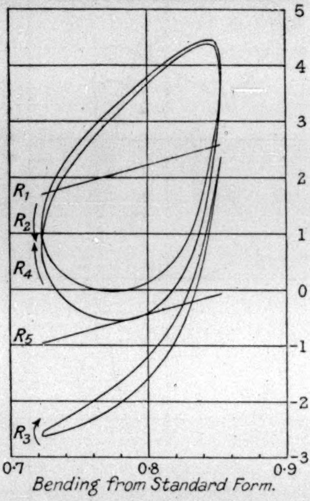

FIG. 8.—CURVATURES OF THE SURFACES OF QUADRUPLE OBJECTIVES WITH FLINT LENS LEADING.

Refractive indices 1·50 and 1·62. Log $\nu/\nu' = 0{\cdot}20$.

For the purpose of comparison with these quadruple objectives, two others have been calculated. The first is the usual form of cemented doublet free from first order spherical aberration but not free from coma. The second is the usual form of astronomical objective with internal surfaces of different curvatures, but satisfying the conditions for freedom from both spherical aberration and coma. These two objectives are made from the same glasses as the quadruple lenses. The chromatic difference of spherical aberration of the former is indicated on Fig. 4 by a small circle, and the chromatic differences of spherical aberration and coma of the latter by small crosses. The best forms of triple and quadruple objectives are evidently about twice as good as either the cemented doublet or the astronomical objective as regards the chromatic difference of spherical aberration. For further comparison,

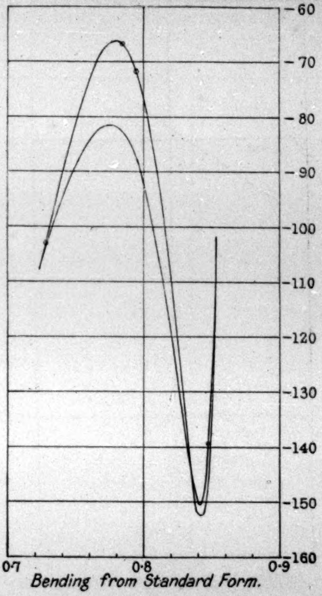

FIG. 9.—SECOND ORDER SPHERICAL ABERRATION FOR OBJECTIVES WITH FLINT LENS LEADING.

including aberrations of order greater than the second, rays have been traced through zones of these objectives, and through the best corrected triple objectives, and also through the quadruple of minimum curvature. The results of these calculations, for which the author is indebted to Miss Dale, Miss Everett and Mr. Trump, are given in Figs. 11 to 15. The curved line shows the second order spherical aberration for various incident heights. The exact values found from tracing through a number of rays are indicated by small crosses. The dots give the corresponding quantities when the refractive indices bear the values used for calculating the chromatic differences in the aberrations. Thus the separation between

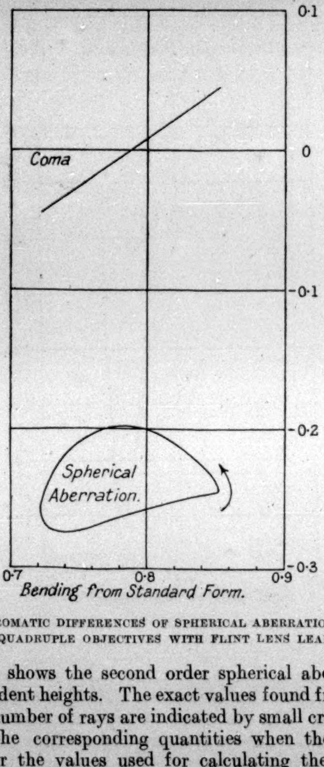

FIG. 10.—CHROMATIC DIFFERENCES OF SPHERICAL ABERRATION AND COMA FOR QUADRUPLE OBJECTIVES WITH FLINT LENS LEADING.

the cross and the curve shows the amount of aberration of orders higher than the second, and the distance between the cross and the dot gives the chromatic difference of aberration. The outstanding features are the relatively bad performance of the astronomical objective and the very small higher order aberrations of the objectives which have small second order aberrations. The conditions satisfied by these lenses are not those which would be selected in use, inasmuch as a compromise between the aberrations of different orders would be preferred. The first order aberrations are entirely removed in each case, and the outstanding errors shown in the diagrams enable the merits of the different forms for this particular pair of glasses—which it will be realised are quite arbitrary and not necessarily realisable—to be compared. The absolute amount of

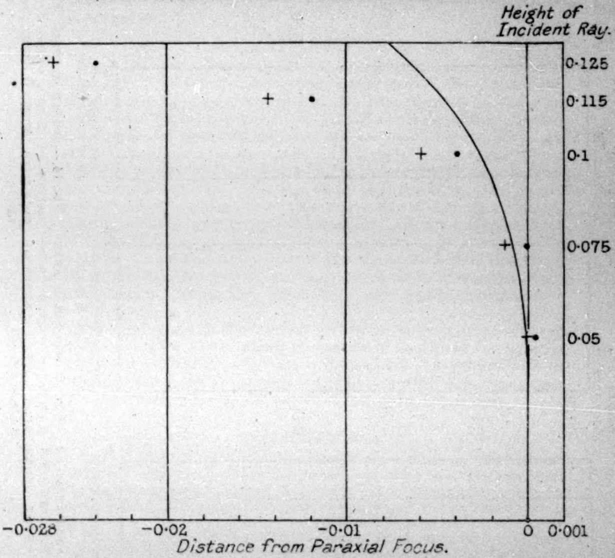

FIG. 11.—SPHERICAL ABERRATION OF UNCEMENTED TELESCOPE OBJECTIVE, USUAL ASTRONOMICAL FORM.

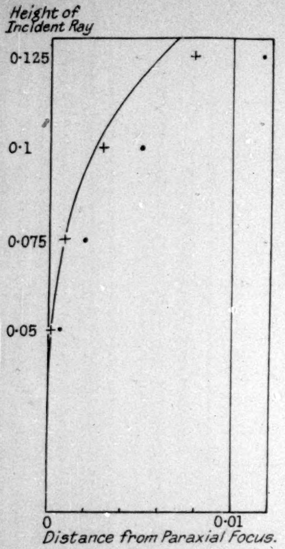

FIG. 12.—SPHERICAL ABERRATION OF CEMENTED DOUBLET WITH CROWN LENS LEADING, FREE FROM FIRST ORDER ABERRATION.

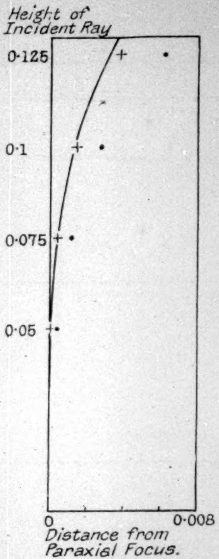

FIG. 13.—SPHERICAL ABERRATION OF TRIPLE OBJECTIVE WITH EXTERNAL CROWN LENSES.

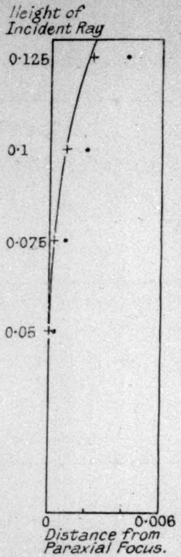

FIG. 14.—SPHERICAL ABERRATION OF TRIPLE OBJECTIVE WITH EXTERNAL FLINT LENSES.

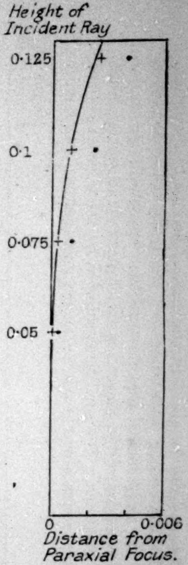

FIG. 15.—SPHERICAL ABERRATION OF QUADRUPLE OBJECTIVE WITH SMALLEST CURVATURES.

spherical aberration for a given colour can be reduced to nearly one eighth of the value shown in these figures for a given maximum aperture by the introduction of a suitable small amount of first order aberration. It would be necessary for such a comparison to be extended over the whole range of possible glasses before the advantages obtainable from the use of four elements in an objective of two glasses could be fully estimated. The calculations described in the present paper show that this class of objective merits more detailed numerical investigation.

ABSTRACT.

The objectives dealt with are cemented combinations of several thin lenses. Two kinds of glass only are employed, the odd elements being of one kind, say, crown, and the even elements of the other kind, flint. Such lenses may be regarded as combinations of achromatic cemented doublets, and formulæ are found for the aberration coefficients of such lenses in terms of those of a standard doublet when the geometrical conditions for the absence of air-gaps between the components are satisfied. Generally speaking, the results reached are that the outer surfaces are concerned with coma, and the internal surfaces with spherical aberration. In all cases the determination of a system to satisfy given conditions involves only the solution of a quadratic equation, and an algebraic method thus effects a solution in a fraction of the time involved in a trigonometrical investigation. Chromatic differences of first order aberrations are easily determined.

The application of the method is illustrated by a series of quadruple objectives which satisfy the ordinary conditions for telescope objectives. Diagrams show the variation of the curvatures with the different forms, the magnitude of the second order spherical aberration, and the chromatic differences of first order aberrations.

DISCUSSION.

Prof. J. W. Nicholson congratulated the author on the increased simplicity which he had brought to some of these important problems. He thought a good case had been made out for the further numerical investigation of this type of objective, and hoped the author would let the Society have the results of such an investigation soon.

Prof. A. E. Conrady (in a communication which was read by the Secretary) said that the Paper could not be regarded as of much practical value. No practical optician would think of constructing a telescope objective of four or more cemented components; even a triple objective is avoided whenever possible, as the technical difficulties are very serious when there are several cemented faces in a lens of considerable size; and he could not conceive of the necessity of such complication ever arising in small lenses for telescopes.

Mr. T. Smith said a very great deal of work would be involved in a systematic exploration of the properties of multiple lenses over the possible range of optical glasses, and the publication of such an extension could not be expected in a short time. He did not quite understand Prof. Conrady's point of view, which would be most deplorable if it were generally adopted by the optical industry; for it could only lead to utter stagnation. Obviously there were always difficulties in making a complex rather than a simple instrument; but difficulties must not be allowed to retard progress, and no investigation should be starved out because of them. Unless investigations are made it is impossible to say whether the advantages they may show are sufficient to compensate the attendant disadvantages. As a matter of fact, the question of employing quadruple objectives had actually arisen as a practical proposition in an instrument designed by a practical optician. In the particular case the optical conditions turned out to be unfavourable to the use of such a lens. Had it been otherwise there is no doubt it would have been employed, and any technical difficulties—which were not regarded as a deciding factor—would without doubt have been overcome. In Germany certainly, and in this country also, he believed, quadruple and even quintuple cemented objectives had been made.

An Exhibition of the Uses of Certain Methods of Classification in Optics was given by Mr. T. H. Blakesley, *M.A., at the Meeting on November 23rd,* 1917.

This consisted of an account of the additions which, in the course of the intervening years, he had been enabled to make in the general diagram of optical properties, first communicated by him to the Physical Society in the year 1903 (" Proceedings," Vol. XVIII., p. 591). The plan pursued is to take as variables the relations which the radii of face curvature bear to the thickness between the faces along the axis. By this means the shape of the lens is given by the two rectangular co-ordinates alone, and any possible property dependent upon a function of these co-ordinates will be represented by a line upon the diagram. When two such loci intersect, the lens corresponding to the points of intersection possesses both the properties corresponding to the lines. A point much dwelt upon by the author was the very large number of straight-line loci corresponding to properties of value in a lens, and of these very many are parallel, and, cutting the axes at 45 deg., may be most simply defined by the value of the intercept of the axes.

It was pointed out that, in general, a lens may have its radii of face curvature both multiplied by the same factor without changing in sign or value the focal length. One of the above-mentioned loci at 45 deg. to the axes represents the only family in which this change cannot be effected, from the fact that the factor in this case is unity. Another of these straight lines belongs to a family in which the two focal lengths corresponding to two assigned indices of refraction are equal; and closely allied to this is a family for which the focal length is a minimum for an assigned value of index.

In another family of the kind the property is that a lens may be immersed in another medium without having its focal length changed.

In another, if a lens is cut out of a cylinder of glass, the remnants of the cylinder in their original position will be achromatic.

In another, telescopic; and so for many others. Other straight lines exist which are not parallel to those above mentioned. They often refer to matters connected with the passage at minimum deviation through a lens, and sometimes to what are called self-conjugate points.

The detection of lens properties which are independent of one of the face curvatures was explained, and some few cases pointed out—*e.g.*, when a lens has one of its radii of face curvature equal to the thickness of the lens at the axis, it matters not what curvature is given to the other face, the point of magnification equal to the index will be coincident with its own conjugate point—*i.e.*, for the point of magnification equal to the inverse of the index for the other side of the lens; and this whichever way the light is passed through the lens.

There are two lines upon the diagram, both straight lines, which refer to the silvering of the second surfaces of lenses, so as to produce plane virtual mirrors; one performs this by sending the centre of the virtual mirror to infinity, the other by sending the surface of the virtual mirror to infinity. In the latter case, which alone calls for special remark, light, though entering the system at an angle, returns upon the same path, always producing an inverted image of -1 magnification, crossing the object at the virtual centre.

DISCUSSION.

Mr. T. **Smith** suggested that the author might add a number of curves to his diagram showing the aberration properties of lenses. There were a number of other geometrical loci that might also be added. It usually happened that the lenses required in actual instruments had too long radii in comparison with the thickness to be included in the region covered by the author's diagram, and it was usually better to calculate each lens by known methods than to extract them from a diagram.

Mr. S. D. **Chalmers** said he had on occasion found diagrams somewhat similar to Mr. Blakesley's, but in which the inverse of the radii of curvatures were employed, to be of considerable service in certain problems.

Mr. **Blakesley** did not think Mr. Chalmers' system would lead to so many straight-line loci as his own. He thought straight-line loci had some advantage if they could be obtained.

V. *Some Problems of Atomic Stability.* *By* Prof. J. W. Nicholson, *M.A., D.Sc., F.R.S.*

Received Aug. 21, 1917.

It is well known that Sir J. J. Thomson's atomic model, in which the electrons are attracted to the centre of the atomic nucleus according to the law of direct distance, admits stable arrangements in which the electrons are not confined to the same plane. The Rutherford model, on the other hand, appears to be incompatible with such arrangements.* Certain special cases, however, are at first sight exceptional, and require a further examination. They are frequently used in the qualitative construction of atoms designed to have special physical properties. An atom of a pyramidal form, for example, consisting of a nucleus, a ring of electrons in the form of a circle whose axis goes through the nucleus, and a single stationary electron on this axis, is a simple case of a type which has been used to a great extent by Stark in his theoretical molecules, in which the linkage binding the atoms together consists of a stationary electron. The discussion of such pyramidal forms is the object of this Paper, in which their dynamical stability is investigated. It is to be pointed out that the investigation proceeds according to the classical dynamical method of treating problems of stability of moving systems, and has no relation to the criterion of stability used in the non-Newtonian mechanics of the atom developed, for example, by Bohr. While this new development is on its trial, it is nevertheless necessary to know where any suggested model stands with reference to the classical dynamics, and some of the constructions suggested by Stark and others have never been examined in any strict quantitative sense. Many are shown to be impossible whichever view of atomic dynamics is adopted.

Atoms of this type appear at first sight to have a possible existence with the solitary electron at rest, and the ring rotating uniformly about the axis, thereby constituting an exception to the statement that all the electrons in a Rutherford model must, according to the principles of Newtonian dynamics, be in the same ring. They are also of considerable interest on account of their unsymmetrical nature, which would involve special properties for one particular electron. It is known that four electrons arranged at the corners of a tetrahedron can revolve in stable steady motion in a Thomson atom, and an analogous possibility might be expected in the present case. So far as radiation is concerned, these pyramidal atoms present no difficulty, for in steady motion of the ring the vector sum of the accelerations of all the charges in the atom is zero, and this is the condition for relatively small radiation.

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* "Phil. Mag.," April, 1914, p. 546; July, 1914, p. 94.

The most fundamental conclusion reached in this paper is that no positively charged or neutral atom can exist in this form. Quite apart from any question of stability, it is not even possible, as may be shown rigorously, to satisfy the ordinary conditions for a steady rotation of the ring in such atoms.

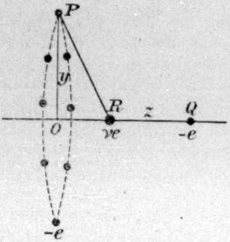

FIG. 1.

PO is the radius $=y$.
P is any electron.
Q is the solitary electron.
R is the nucleus.
RQ is the atomic axis.

This conclusion appears to remove the possibility of molecular structures of the type already mentioned. Moreover, even though an atom may theoretically take up an electron on its axis, if it is neutral or negatively charged before the operation, the ensuing structure is unstable as regards some of its more important vibrations. Its stability in fact is not comparable with that of a single ring, and it could not be endowed with any permanence. The vibrations which it can sustain, however, during a transitory existence, are of considerable interest and several numerical cases are worked out in detail. The direct application of Earnshaw's theorem of stability is not very helpful in atomic theory, in that unstable modes are, in general, those least readily excited.

We shall investigate, in the first place, a general pyramidal atom, in which there are n electrons in the base forming a ring and a single electron at the vertex, the nucleus νe being situated on the axis. Let y be the radius of the ring, x its distance from the nucleus, z the distance of the solitary electron from the nucleus, and m the mass of an electron. The orientation of an electron in the plane of the ring may be denoted by θ. The arrangement is indicated in the annexed figure.

The Lagrangian method supplies the readiest means of solution for the principal vibrations. The kinetic and potential energies are given by

$$2\tau = m\dot{z}^2 + mn(\dot{x}^2 + \dot{y}^2 + y^2\dot{\theta}^2),$$

$$V = -\frac{\nu e^2}{z} + \frac{ne^2}{\sqrt{y^2+(x+z)^2}} + \frac{ne^2 S_n}{4y} - \frac{\nu ne^2}{\sqrt{x^2+y^2}},$$

when the displacements are such as to retain a plane ring—the important case corresponding to vibrations of "class zero," as defined in previous investigations.* S_n has the usual meaning,

$$S_n = \sum_{r=1}^{n-1} \operatorname{cosec} \frac{r\pi}{n}.$$

The equations of motion become

$$m\ddot{z} = -\frac{\nu e^2}{z^2} + \frac{ne^2(x+z)}{\{y^2+(x+z)^2\}^{\frac{3}{2}}}.$$

$$m\ddot{x} = -\frac{\nu e^2 x}{(x^2+y^2)^{\frac{3}{2}}} + \frac{e^2(x+z)}{[y^2+(x+z)^2]^{\frac{3}{2}}},$$

$$\frac{d}{dt}(y^2\dot{\theta}) = 0.$$

$$m(\ddot{y} - y\dot{\theta}^2) = \frac{e^2 S_n}{4y^2} - \frac{\nu e^2 y}{(x^2+y^2)^{\frac{3}{2}}} + \frac{e^2 y}{[y^2+(x+z)^2]^{\frac{3}{2}}}.$$

We have regarded the nucleus as fixed during the vibration. The knowledge of its effect on periods, already possessed, shows that it cannot alter the character of the solution, or even the numerical values to more than a few tenth metres, in the case of wave-lengths of emitted radiation. The error is in fact of order m/M, where M is the mass of the nucleus.

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* "Monthly Notices" of R.A.S., Nov., 1911, *et seq.*

The equations of steady motion, derived by equating accelerations to zero, are reduced to

$$(x+z)z^2=\frac{\nu}{n}[y^2+(x+z)^2]^{\frac{3}{2}},$$

$$(x^2+y^2)^{\frac{3}{2}}=nxz^2,$$

$$my\omega^2=-\frac{e^2S_n}{4y^2}+\frac{\nu e^2y}{(x^2+y^2)^{\frac{3}{2}}}-\frac{e^2y}{[y^2+(x+z)^2]^{\frac{3}{2}}}.$$

The first pair serve to determine the ratios $x:y:z$, and do not involve ω. the steady angular velocity. Thus the structure of such an atom would be definite, in the geometrical sense, and a change in the angular velocity would only cause an expansion or contraction of all its distances in the same ratio. The relative distances of the atom in fact are determined solely by n and ν.

For the examination of the vibrations about steady motion, we may write $(x+x_1\ y+y_1\ z+z_1\ \omega+\dot{\phi})$ for $(x,\ y,\ z,\ \dot{\theta})$, so that (x, y, z) now denote steady motion values, and the squares of x_1, y_1, z_1, and $\dot{\phi}$ may be neglected in the usual manner. Then

$$m\ddot{z}_1=-V_{zx}x_1-V_{zy}y_1-V_{zz}z_1,$$

$$mn\ddot{x}_1=-V_{xx}x_1-V_{xy}y_1-V_{xz}z_1,$$

$$mn(\ddot{y}_1-\omega^2y_1-2\omega y\dot{\phi})=-V_{xy}x_1-V_{yy}y_1-V_{yz}z_1,$$

$$y^2\dot{\phi}+2\omega yy_1=0,$$

the suffixes to V denoting differentiations. Thus, if all the variations are proportional to e^{iqt}, in a periodic variation from the steady motion,

$$mq^2z_1\quad=V_{zx}x_1+V_{zy}y_1+V_{zz}z_1,$$

$$mnq^2x_1=V_{xx}x_1+V_{xy}y_1+V_{xz}z_1,$$

$$mn(q^2-3\omega^2)y_1=V_{xy}x_1+V_{yy}y_1+V_{yz}z_1,$$

with the period equation

$$\begin{vmatrix} q^2-V_{xx}/mn & -V_{xy}/mn & -V_{xz}/mn \\ -V_{xy}/mn & q^2-3\omega^2-V_{yy}/mn & -V_{yz}/mn \\ -V_{xz}/mn & -V_{yz}/mn & \frac{q^2}{n}-V_{zz}/mn \end{vmatrix}=0,$$

which is a cubic. There are three principal vibrations of class zero, and the three values of q^2 must all be real and positive if the stability of the system is to be comparable with that of a simple ring, which, moving as a whole in vibrations of class zero, is completely stable.

Let $$x/y=\alpha, \quad z/y=\beta.$$

Then the conditions of steady motion become

$$(1+\alpha^2)^{\frac{3}{2}}=n\alpha\beta^2,$$

$$\nu[1+(\alpha+\beta)^2]^{\frac{3}{2}}=n\beta^2(\alpha+\beta),$$

$$\begin{aligned} my^3\omega^2/e^2 &= -\frac{1}{4}S_n+\frac{\nu}{(1+\alpha^2)^{\frac{3}{2}}}-\frac{1}{[1+(\alpha+\beta)^2]^{\frac{3}{2}}} \\ &= -\frac{1}{4}S_n+\frac{\nu}{n\alpha\beta^2}-\frac{\nu}{n\beta^2(\alpha+\beta)} \\ &= -\frac{1}{4}S_n+\frac{\nu}{n\alpha\beta(\alpha+\beta)}, \end{aligned}$$

substituting from the first two conditions in order to satisfy the third. This third condition relates the angular velocity and radius of the ring in the usual manner, and the first two conditions serve to determine the shape of the atom.

We find the following values for the derivates of V :—

$$-\frac{1}{mn}V_{yz}=-\frac{3\nu e^2}{mny^3}\cdot\frac{\alpha+\beta}{[1+(\alpha+\beta)^2]^{\frac{5}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

$$=-\frac{3\nu e^2}{mn^2y^3\beta^2}\cdot\frac{1}{1+(\alpha+\beta)^2}$$

$$-\frac{1}{mn}V_{zx}=\frac{e^2\nu^2}{mn^2y^3\beta^2(\alpha+\beta)}\cdot\frac{1-2(\alpha+\beta)^2}{1+(\alpha+\beta)^2} \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

$$-\frac{1}{mn}V_{zz}=\frac{\nu e^2}{mny^3\beta^3}\left\{2+\frac{\beta}{\alpha+\beta}\cdot\frac{1-2(\alpha+\beta)^2}{1+(\alpha+\beta)^2}\right\} \quad . \quad . \quad . \quad . \qquad (3)$$

$$-\frac{1}{mn}V_{xy}=\frac{3\nu e^2}{mny^3\beta^2}\left\{\frac{1}{1+\alpha^2}-\frac{1}{1+(\alpha+\beta)^2}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

$$-\frac{1}{mn}V_{xx}=\frac{\nu e^2}{mny^3\beta^2}\left\{\frac{1-2(\alpha+\beta)^2}{(\alpha+\beta)\{1+(\alpha+\beta)^2\}}-\frac{1-2\alpha^2}{\alpha(1+\alpha^2)}\right\} \quad . \qquad (5)$$

$$-\frac{V_{yy}}{mn}=\frac{e^2}{my^3}\left\{\frac{1}{2}S_n-\frac{\nu(\alpha^2-2)}{n\alpha\beta^2(1+\alpha^2)}+\frac{\nu}{n\beta^2(\alpha+\beta)}\,\frac{(\alpha+\beta)^2-2}{1+(\alpha+\beta)^2}\right\} \quad . \qquad (6)$$

from which the period equation may be expressed in general terms as an equation in q/ω with numerical co-efficients, when (α, β) are found.

The Possible Steady Motions.

The ratios (α, β) in general satisfy

$$(1+\alpha^2)^{\frac{3}{2}}=n\alpha\beta^2,$$

$$\nu[1+(\alpha+\beta)^2]^{\frac{3}{2}}=n(\alpha+\beta)\beta^2.$$

Write $\alpha=\tan\theta$, so that

$$\beta^2=\frac{1}{n\sin\theta\cos^2\theta}, \quad \beta=\frac{1}{\cos\theta\sqrt{n\sin\theta}}.$$

Therefore θ satisfies

$$\left\{1+\left(\tan\theta+\frac{\sec\theta}{\sqrt{n\sin\theta}}\right)^2\right\}^3=\frac{n^2}{\nu^2}\cdot\frac{1}{n^2\sin^2\theta\cos^4\theta}\left(\tan\theta+\frac{\sec\theta}{\sqrt{n\sin\theta}}\right)^2,$$

or, on reduction,

$$[n\sin\theta\cos^2\theta+(1+\sqrt{n}\sin^{\frac{3}{2}}\theta)^2]^3=\frac{n^2}{\nu^2}(1+\sqrt{n}\sin^{\frac{3}{2}}\theta)^2.$$

Since by their nature, α and β are positive, the necessary value of θ must lie between 0 and $\frac{\pi}{2}$. It is evident that when n is less than ν, no such value of θ can satisfy the equation, so that *a pyramidal form of a positively charged atom is not possible.* The same result applies to a neutral atom, with $n=\nu-1$.

When the atom has a single negative charge, $n=\nu$, and the solution is $\theta=0$, or $\alpha=0$, $\beta=\infty$, indicating the single-ring type of a neutral atom, since the extra electron is at an infinite distance. But real values of α and β are possible for an atom with at least one negative charge. We shall work out in detail the case of a nucleus of strength $5e$, with seven electrons, six of them being in a ring.

Writing $n\sin\theta=w^2$, the new magnitude w is determined by

$$\left[w^2\left(1-\frac{w^4}{n^2}\right)+\left(1+\frac{w^3}{n}\right)^2\right]^3=\frac{n^2}{\nu^2}\left(1+\frac{w^3}{n}\right)^2,$$

or

$$\left(1+w^2+2\frac{w^3}{n}\right)^3=\frac{n^2}{\nu^2}\left(1+\frac{w^3}{n}\right)^2,$$

where, in terms of w,

$$\alpha=w^2/(n^2-w^4)^{\frac{1}{2}}, \quad \beta=n/w(n^2-w^4)^{\frac{1}{2}}.$$

In the case of the suggested illustration, $n=6$, $\nu=5$, and

$$\left(1+w^2+\frac{1}{3}w^3\right)^3=\frac{16}{25}\left(1+\frac{w^3}{6}\right)^2.$$

This equation has a root between zero and unity, but no other real root. Thus, as would be expected, only one such arrangement is possible. The single-ring arrangement corresponds to a neglected factor $w=0$.

The equation can be solved by continued approximation, and the value of the real root, to five significant figures, is

$$w=0{\cdot}34721,$$

leading to

$$\alpha=0{\cdot}020096,$$
$$\beta=2{\cdot}8807, \qquad \alpha+\beta=2{\cdot}9008,$$

and thence,

$$\omega^2=3{\cdot}1349\frac{e^2}{my^3},$$

showing that a steady motion is possible. The following are then obtained after some reduction :

$$V_{yz}=0{\cdot}008506mn\omega^2, \qquad V_{zx}=0{\cdot}015472mn\omega^2,$$
$$V_{zz}=-0{\cdot}003674mn\omega^2, \qquad V_{xy}=-0{\cdot}08585mn\omega^2,$$
$$V_{xx}=1{\cdot}6112mn\omega^2, \qquad V_{yy}=-4{\cdot}3599mn\omega^2,$$

and the period equation is

$$\begin{vmatrix} q^2/\omega^2-1{\cdot}6112 & 0{\cdot}08585 & -0{\cdot}01547 \\ 0{\cdot}08585 & q^2/\omega^2+1{\cdot}3599 & -0{\cdot}00851 \\ -0{\cdot}01547 & -0{\cdot}00851 & q^2/\omega^2+0{\cdot}00367 \end{vmatrix}=0,$$

or on expansion, if $\rho=q^2/\omega^2$,

$$\rho^3-0{\cdot}2293\rho^2-2{\cdot}20584\rho-0{\cdot}04974=0.$$

This equation has a positive root greater than unity, and two negative roots indicating unstable modes. The vibrations now investigated are those of class zero—the simplest vibrations, and those set up most readily—and the fact that two are unstable shows that the atom could only have a very transitory existence.

The positive root is found to be

$$q^2/\omega^2=1{\cdot}6145,$$

or

$$q/\omega=1{\cdot}2706,$$

which, of course, needs a small correction, arising from the fact that the positive nucleus of the atom moves slightly during a vibration.

We may notice that the unstable modes have no periodic feature, for q/ω is in each of them purely imaginary, and the corresponding motion in these modes is a continued enlargement of the atom in all its dimensions, whose ratios remain constant.

The Energy of a Pyramidal System.

At this point we must notice an important formula for the energy of a system of the present type. It is, in fact, a special case of a much more general theorem relating to systems of charges in steady motion under the inverse square law. We do not, however, in the present Paper, discuss the more general significance of the energy relation, as it is not directly relevant. The energy of the present system may be written in the form

$$T+V=\frac{1}{2}mny^2\omega^2+\frac{e^2}{y}\left\{-\frac{\nu}{\beta}+\frac{n}{[1+(\alpha+\beta)^2]^{\frac{1}{2}}}+\frac{1}{4}nS_n-\frac{n\nu}{(1+\alpha^2)^{\frac{1}{2}}}\right\},$$

or by virtue of the equations of steady motion,

$$\begin{aligned}T+V&=\frac{1}{2}mny^2\omega^2+\frac{e^2}{y}\left\{-\frac{\nu}{\beta}+\frac{\nu}{\beta^2}\cdot\frac{1+(\alpha+\beta)^2}{\alpha+\beta}+\frac{nS_n}{4}-\frac{\nu}{\alpha\beta^2}(1+\alpha^2)\right\}\\&=\frac{1}{2}mny^2\omega^2+\frac{e^2}{y}\left\{\frac{1}{4}nS_n+\frac{\nu}{\beta^2}\left(\frac{1}{\alpha+\beta}-\frac{1}{\alpha}\right)\right\}\\&=\frac{1}{2}mny^2\omega^2-\frac{ne^2}{y}\left\{-\frac{1}{4}S_n-\frac{\nu}{n\alpha\beta(\alpha+\beta)}\right\}\\&=-\frac{1}{2}mny^2\omega^2.\end{aligned}$$

To this a constant C must be added, being the energy of the system when scattered to a state of infinite dispersion. It is remarkable that the formula for the energy is of the same type as for a single ring, in terms of radius and angular velocity.

Thus for n electrons in a ring of radius a, the energy is

$$C-\frac{1}{2}mna^2\omega^2,$$

and the same formula is valid for the total energy if there is also a stationary electron on the axis, and, as will appear later, if there are two such stationary electrons, one on either side of the axis, thereby forming a symmetrical atom.

A Symmetrical Atom with Two Stationary Electrons.

In order to complete the investigation, one further problem must be considered, relating to an atom which at first sight also appears to contain a greater possibility of limited stability, combined with symmetry.

We therefore next take up the case, of which the analysis is simpler, of a ring of n electrons with a solitary electron on *both* sides.

Attention may be restricted to the case in which the electrons are always equidistant from the nucleus, so that the ring does not vibrate transversely. If y is its radius and x the distance in the figure, the Lagrangian function for such vibrations is given by

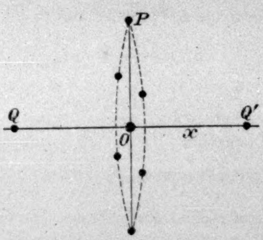

FIG. 2.

O is the nucleus.
Q, Q' the solitary electrons.
P an electron of the ring.

PO = radius = y.
$OQ = OQ' = x$.

$$2T = mn(\dot{y}^2 + y^2\dot{\theta}^2) + 2m\dot{x}^2,$$

$$V = -\left(2\nu - \frac{1}{2}\right)\frac{e^2}{x} + \frac{2ne^2}{\sqrt{x^2+y^2}} - \frac{ne^2}{y}\left(\nu - \frac{1}{4}S_n\right),$$

and

$$2m\ddot{x} = -\frac{\partial V}{\partial x}$$

$$mn(\ddot{y} - y\dot{\delta}^2) = -\frac{\partial V}{\partial y}$$

$$\frac{d}{dt}(y^2\theta) = 0.$$

The conditions of steady motion are

$$\nu - \frac{1}{4} = \frac{nx^3}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\omega^2 = \frac{e^2}{my^3}\left\{\nu - \frac{1}{4}S_n - \frac{2y^3}{(x^2+y^2)^{\frac{3}{2}}}\right\}.$$

For the small vibration, writing as before

$$\dot{\theta}=\omega+\phi, \quad \text{and} \quad x+x_1, \quad y+y_1, \quad \text{for } (x, y),$$

so that (x, y) are the steady motion values,

$$2m\ddot{x}_1=-x_1V_{xx}-y_1V_{xy},$$

$$mn(\ddot{y}_1+3\omega^2y_1)=-x_1V_{xy}-y_1V_{yy},$$

or if the displacements are proportional to e^{iqt},

$$x_1\left(q^2-\frac{V_{xx}}{2m}\right)=y_1\frac{V_{xy}}{2m}$$

$$y_1\left(q^2-3\omega^2-\frac{V_{yy}}{mn}\right)=x_1\frac{V_{xy}}{mn},$$

giving a period equation

$$\left(q^2-\frac{V_{xx}}{2m}\right)\left(q^2-3\omega^2-\frac{V_{yy}}{mn}\right)=\frac{1}{2n}\left(\frac{V_{xy}}{m}\right)^2,$$

with two principal modes of vibration. We find

$$V_{xx}=-(4\nu-1)\frac{e^2}{x^3}-\frac{2ne^2(y^2-2x^2)}{(x^2+y^2)^{\frac{5}{2}}}=\frac{-6ne^2y^2}{(x^2+y^2)^{\frac{5}{2}}},$$

$$V_{yy}=-\frac{2ne^2}{y^3}\left(\nu-\frac{1}{4}S_n\right)-\frac{2ne^2(x^2-2y^2)}{(x^2+y^2)^{\frac{5}{2}}}=-2nm\omega^2-\frac{6ne^2x^2}{(x^2+y^2)^{\frac{5}{2}}},$$

$$V_{xy}=\frac{6ne^2xy}{(x^2+y^2)^{\frac{5}{2}}}.$$

Let $$\sigma=\frac{x}{y}=\left[\left(\frac{\nu}{n}-\frac{1}{4n}\right)^{-\frac{2}{3}}1-\right]^{-\frac{1}{2}}.$$

Then

$$\omega^2=\frac{e^2}{my^3}\left\{\nu-\frac{1}{4}S_n-\frac{2}{(1+\sigma^2)^{\frac{3}{2}}}\right\},$$

$$\frac{V_{xx}}{2m}=-\frac{e^2}{my^3}\left\{\left(2\nu-\frac{1}{2}\right)\frac{1}{\sigma^3}+\frac{n(1-2\sigma^2)}{(1+\sigma^2)^{\frac{5}{2}}}\right\}=-\frac{3n}{(1+\sigma^2)^{\frac{5}{2}}}\frac{e^2}{my^3},$$

$$\frac{V_{yy}}{mn}=-\frac{e^2}{my^3}\left\{2\left(\nu-\frac{1}{4}S_n\right)+\frac{2(\sigma^2-2)}{(1+\sigma^2)^{\frac{5}{2}}}\right\}=-2\omega^2-\frac{6\sigma^2}{(1+\sigma^2)^{\frac{5}{2}}}\frac{e^2}{my^3},$$

$$\frac{V_{xy}}{m}=\frac{e^2}{my^3}\left\{\frac{6n\sigma}{(1+\sigma^2)^{\frac{5}{2}}}\right\}.$$

In this type of vibration the positive nucleus does not move and calculations of vibrations for this system are free from the unusual complexity of the former. It is evident that no steady motion exists for atoms of the present type, if they have a positive charge.

We shall calculate the case $\nu=5$, $n=6$. Thus

$$\sigma=[(0\cdot79167)^{-\frac{2}{3}}-1]^{-\frac{1}{2}}=(0\cdot16852)^{-\frac{1}{2}}=2\cdot4359$$

$$(1+\sigma^2)^{-\frac{3}{2}}=0\cdot054770,$$

$$\omega^2=\frac{e^2}{my^3}\{3\cdot0631\},$$

$$\frac{V_{xx}}{2m}=-\omega^2(0\cdot046418),$$

$$\frac{V_{yy}}{mn}=-\omega^2(2\cdot091812),$$

$$\frac{V_{xy}}{m}=\omega^2(0\cdot22614), \quad \frac{1}{2n}\left(\frac{V_{xy}}{m}\right)^2=0\cdot004262,$$

and the period equation is

$$\left(\frac{q^2}{\omega^2}+0\cdot04642\right)\left(\frac{q^2}{\omega^2}-0\cdot90819\right)=0\cdot004262,$$

with a positive and a negative root. The negative root does not correspond to a periodic disturbance, and the system is unstable. The value of the positive root is ultimately

$$q^2/\omega^2=0\cdot91263,$$

or

$$q/\omega=0\cdot95532.$$

The 7-ring with Two Solitary Electrons.

Before commenting further on the vibrations peculiar to the system last dealt with, we proceed to collect numerical data of sufficient completeness to make possible a general view of these systems. For this purpose we perform a similar calculation with seven electrons in the ring. The stages of the calculation are set forth below, with $n=7$, $\nu=5$.

$$\sigma=\left[\left(\frac{\nu}{n}-\frac{1}{4n}\right)^{-\frac{2}{3}}-1\right]^{-\frac{1}{2}}=(0\cdot29500)^{-\frac{1}{2}}=1\cdot84115,$$

$$1+\sigma^2=4\cdot38983,$$

$$\omega^2=\frac{e^2}{my^3}\left(2\cdot6952-\frac{2}{(1+\sigma^2)^{\frac{3}{2}}}\right)=\frac{e^2}{my^3}(2\cdot4778),$$

$$\frac{V_{xx}}{2m}=-\omega^2(0\cdot20991),$$

$$\frac{V_{yy}}{2m}=-\omega^2(2\cdot203305),$$

$$\frac{V_{xy}}{m}=\omega^2(0\cdot77295), \quad \frac{1}{2n}\left(\frac{V_{xy}}{m}\right)^2=0\cdot042675,$$

and the period equation is, with $\rho = q^2/\omega^2$,

$$(\rho + 0{\cdot}20991)(\rho - 0{\cdot}796695) = 0{\cdot}042675,$$

or

$$\rho^2 - 0{\cdot}58678\ \rho - 0{\cdot}20991 = 0\,;$$

whence

$$\rho = 0{\cdot}83743,$$

or

$$q/\omega = 0{\cdot}9151$$

for the real root. There is an imaginary root as before.

The 8-Ring with Two Solitary Electrons.

In this case,

$$\sigma = [(0{\cdot}59375)^{-\frac{2}{3}} - 1]^{-\frac{1}{2}} = (0{\cdot}41557)^{-\frac{1}{2}} = 1{\cdot}55123,$$

$$1 + \sigma^2 = 3{\cdot}4063,$$

$$\omega^2 = \frac{e^2}{my^3}\left(2{\cdot}1951 - \frac{2}{(1+\sigma^2)^{\frac{3}{2}}}\right) = \frac{e^2}{my^3}(1{\cdot}8770),$$

$$\frac{V_{xx}}{2m} = -\omega^2(0{\cdot}59709),$$

$$\frac{V_{yy}}{mn} = -\omega^2(2{\cdot}35920),$$

$$\frac{V_{xy}}{m} = \omega^2(1{\cdot}8525), \quad \frac{1}{2n}\left(\frac{V_{xy}}{m}\right)^2 = 0{\cdot}214475,$$

and if

$$\rho = q^2/\omega^2,$$

$$(\rho + 0{\cdot}59709)(\rho - 0{\cdot}64080) = 0{\cdot}21447,$$

or

$$\rho^2 - 0{\cdot}04371\ \rho - 0{\cdot}59709 = 0,$$

with a positive root

$$\rho = 0{\cdot}79490,$$

or

$$q/\omega = 0{\cdot}8916,$$

and the usual unstable vibration.

The Energy of the Atom.

We can show that the energy of an atom with a stationary electron on either side of the ring is still given by the same type of formula, as with an electron on one side only, or with a single ring only. For the energy of the atom with two stationary electrons, and n rotating electrons in the ring is, with the previous notation,

$$T + V = \frac{1}{2}mn\omega^2y^2 + \frac{e^2}{y}\left\{\frac{1}{2\sigma} - n\left(\nu - \frac{1}{4}S_n\right) + \frac{2n}{\sqrt{1+\sigma^2}} - \frac{2\nu}{\sigma}\right\},$$

where

$$\nu - \frac{1}{4} = n\sigma^3/(1+\sigma^2)^{\frac{3}{2}}.$$

Thus

$$T+V=\frac{1}{2}mn\omega^2y^2+\frac{e^2}{y}\left\{\frac{1}{2\sigma}-n\left(\nu-\frac{1}{4}S_n\right)-\frac{2\nu}{\sigma}+\frac{2}{\sigma^3}\left(v-\frac{1}{4}\right)(1+\sigma^2)\right\},$$

$$=\frac{1}{2}mn\omega^2y^2+\frac{e^2}{y}\left\{\frac{2}{\sigma^3}\left(\nu-\frac{1}{4}\right)-n\left(\nu-\frac{1}{4}S_n\right)\right\},$$

$$=\frac{1}{2}mn\omega^2y^2-\frac{ne^2}{y}\left\{\nu-\frac{1}{4}S_n-\frac{2}{(1+\sigma^2)^{\frac{3}{2}}}\right\},$$

$$=-\frac{1}{2}mn\omega^2y^2,$$

by the conditions of steady motion. We add the constant C dependent on the zero of measurement, and obtain the energy

as
$$C-\frac{1}{2}mn\omega^2y^2,$$

a formula again of the same type as for the single ring of radius y and angular velocity ω.

If the stable periods actually are periods of emitted radiation, and if radiation takes place in quanta associated with the angular momentum of the ring, it is known that the principal wave-lengths to which they give rise form a series with constant difference of (wave-length)$^{\frac{1}{2}}$, in the case of a single ring. Such series have been detected in the solar coronal spectrum.* If a system of the present kind is also subject to such laws, its stable vibrations would give a spectrum of the same type; but the greater instability of the system would probably preclude its detection.

Transverse Vibrations of the Ring.

It is evident that the vibrations already dealt with in the atoms with two solitary electrons correspond to the simplest vibration in the plane, in the case of a single ring. For the rings of these atoms have only moved in their own planes hitherto. It is necessary to determine the vibration, also of class zero, which corresponds to the simplest transverse vibration of a simple-ring atom—in this case a vibration in which the ring moves transversely, while the displacements of all the electrons of the ring are equal at any instant. If the ring is displaced by an amount s, as in the figure, and (x, z) are the distances of the solitary electrons from the nucleus, at present regarded as stationary, we may write

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* "Monthly Notices" of R.A.S., Supp. No. 1912, p. 730, *et seq.*

$$2T = mn(\dot{y}^2 + y^2\dot{\theta}^2 + \dot{s}^2) + m(\dot{x}^2 + \dot{z}^2),$$

$$V = -\nu e^2\left(\frac{1}{x} + \frac{1}{z}\right) + \frac{e^2}{x+z} + ne^2\left\{\frac{1}{\sqrt{y^2 + (x-s)^2}} + \frac{1}{\sqrt{y^2 + (z+s)^2}}\right\}$$
$$+ \frac{ne^2 s_n}{4y} - \frac{ne^2}{\sqrt{y^2 + s^2}},$$

where in the steady motion, $x = z$, $s = 0$.

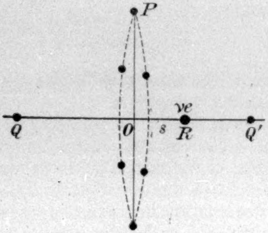

FIG. 3.

$OQ = x$.
$OQ' = z = x - s$.
$OR = s$.
$OP = \text{radius} = y$.

The equations of motion for the co-ordinates (x, z, s) are

$$m\ddot{x} = -V_x, \quad m\ddot{z} = -V_z, \quad m\ddot{s} = -V_x,$$

or if $x + x_1$, $z + z_1$ are put for (x, z), which now denote steady motion values, we derive

$$mq^2 x_1 = x_1 V_{xx} + y_1 V_{xy} + z_1 V_{xz} + s_1 V_{xs},$$
$$mq^2 y_1 = x_1 V_{xy} + y_1 V_{yy} + z_1 V_{zy} + s_1 V_{ys},$$
$$mq^2 s_1 = x_1 V_{sx} + y_1 V_{sy} + z_1 V_{sz} + s_1 V_{ss},$$

where $x = z$, $s = 0$ after differentiation.

There are relations among these coefficients. For example, V_{sy} is given by

$$V_{sy}=\frac{\partial}{\partial y}\left\{\frac{ne^2 s}{(y^2+s^2)^{\frac{3}{2}}}-\frac{(s-x)ne^2}{[y^2+(s-x)^2]^{\frac{3}{2}}}-\frac{(s+z)ne^2}{(y^2+(z+s)^2)^{\frac{3}{2}}}\right\},$$

$$=\frac{\partial}{\partial y}\{0\}=0,$$

when $s=0$, $x=z$.
The last equation is therefore independent of y_1.

Moreover, when s vanishes, V is symmetrical in x and z, and

$$V_{xx}=V_{zz}.$$

We can also show that

$$V_{xs}=-V_{zs}, \quad V_{ss}=-2V_{xs}, \quad V_{xy}=V_{zy},$$

after steady motion values are substituted. The equations, therefore, take the form

$$x_1(mq^2-V_{xx})=y_1V_{xy}+z_1V_{xz}+s_1V_{xs},$$

$$z_1(mq^2-V_{xx})=y_1V_{xy}+x_1V_{xz}-s_1V_{xs},$$

$$s_1(mnq^2+2V_{xs})=(x_1-z_1)V_{xs},$$

or, with the elimination of y_1,

$$(x_1-z_1)(mq^2-V_{xx}+V_{xz})=2s_1V_{xs},$$

$$(x_1-z_1)V_{xs}=s_1(mnq^2+2V_{xs}),$$

with the solution

$$x_1=z_1, \quad s_1=0$$

(which is the vibration already discussed), or

$$(mq^2-V_{xx}+V_{xz})(mnq^2+2V_{xs})=2(V_{xs})^2.$$

This is the period equation of the transverse vibrations of class zero, corresponding to the simple transverse vibrations of a single ring.

With steady motion values,

$$V_{xz}=\frac{e^2}{4x^3}, \quad V_{xx}=-\frac{2\nu e^2}{x^3}+\frac{e^2}{4x^3}-ne^2\left\{\frac{1}{(x^2+y^2)^{\frac{3}{2}}}-\frac{3x^2}{(x^2+y^2)^{\frac{5}{2}}}\right\},$$

$$V_{xs}=ne^2\left\{\frac{1}{(x^2+y^2)^{\frac{3}{2}}}-\frac{3x^2}{(x^2+y^2)^{\frac{5}{2}}}\right\},$$

or with $x/y=\sigma$ as before,

$$\frac{V_{xs}}{mn}=\frac{e^2}{my^3}\left\{\frac{1}{(1+\sigma^2)^{\frac{3}{2}}}-\frac{3\sigma^2}{(1+\sigma^2)^{\frac{5}{2}}}\right\},$$

$$\frac{V_{xz}-V_{xx}}{m}=\frac{e^2}{my^3}\left\{\frac{2\nu}{\sigma^3}+\frac{n}{(1+\sigma^2)^{\frac{3}{2}}}-\frac{3n\sigma^2}{(1+\sigma^2)^{\frac{5}{2}}}\right\},$$

with the period equation

$$\left(q^2+\frac{V_{xz}-V_{xx}}{m}\right)\left(q^2+\frac{2V_{xs}}{mn}\right)=2n\left(\frac{V_{xs}}{mn}\right)^2.$$

Numerical Values.

Proceeding to the numerical calculation of the various systems, we obtain the following results :—

(1) For a ring of six electrons, with $\nu=5$,

$$\sigma=2{\cdot}4359, \qquad \omega^2=3{\cdot}0631\frac{e^2}{my^3},$$

$$V_{xs}=-0{\cdot}028026\, mn\omega^2.$$

$$V_{xz}-V_{xx}=0{\cdot}05771 m\omega^2, \qquad 2n\left(\frac{V_{xs}}{mn}\right)^2=0{\cdot}009425,$$

$$(q^2/\omega^2+0{\cdot}05771)(q^2/\omega^2-0{\cdot}05605)=0{\cdot}009425,$$

with a root

$$q^2/\omega^2=0{\cdot}11212,$$

so that

$$q/\omega=0{\cdot}3348,$$

and a negative root.

(2) For a ring of seven electrons, with $\nu=5$,

$$\sigma=1{\cdot}84115, \qquad \omega^2=2{\cdot}4778\frac{e^2}{my^3},$$

$$V_{xs}=-0{\cdot}05777 mn\omega^2,$$

$$V_{xz}-V_{xx}=0{\cdot}24225 m\omega^2, \qquad 2n\left(\frac{V_{xs}}{mn}\right)^2=0{\cdot}04672,$$

$$(q^2/\omega^2+0{\cdot}24225)(q^2/\omega^2-0{\cdot}11554)=0{\cdot}04672,$$

$$q^2/\omega^2=0{\cdot}21722, \qquad q/\omega=0{\cdot}4661.$$

(3) For a ring of eight electrons, with $\nu=5$,

$$\sigma=1{\cdot}55123, \qquad \omega^2=1{\cdot}8770\frac{e^2}{my^3},$$

$$V_{xs}=-0{\cdot}09485 mn\omega^2,$$

$$V_{xz}-V_{xx}=1{\cdot}5248 m\omega^2, \quad 2n\left(\frac{V_{xs}}{mn}\right)^2=0{\cdot}14394,$$

$$(q^2/\omega^2+1{\cdot}5248)(q^2/\omega^2-0{\cdot}1897)=0{\cdot}14394,$$

$$q^2/\omega^2=0{\cdot}2699, \qquad q/\omega=0{\cdot}5195.$$

Collecting the results, the following vibrations are possible, with $\nu=5$:—

(1) $n=6, \quad \omega^2=3\cdot0631\frac{e^2}{my^3}, \quad q/\omega=0\cdot9553, \quad 0\cdot3348.$

(2) $n=7, \quad \omega^2=2\cdot4778\frac{e^2}{my^3}, \quad q/\omega=0\cdot9151, \quad 0\cdot4661.$

(3) $n=8, \quad \omega^2=1\cdot8770\frac{e^2}{my^3}, \quad q/\omega=0\cdot8916, \quad 0\cdot5195.$

And with them are associated several unstable modes of equal importance. The systems of this type are, therefore, essentially unstable to a degree quite transcending that of simple-ring atoms. They cannot well form part of any molecular structure, and without the necessity of more formal and general investigations, we may conclude that conceptions of molecules such as those of Stark cannot survive the test of quantitative treatment on an ordinary dynamical basis. They may nevertheless be possible under the rule of a non-Newtonian dynamics, but the criticism to which they are open, even in this case, is that, from a consideration of steady motions alone—in which the two varieties of dynamics are identical, as in Bohr's method—the individual atoms can only exist when negatively charged, so that no two of them could well form an electrically neutral molecule.

General Remarks.

Earnshaw's theorem regarding systems of electric charges indicates that atoms of the type contemplated in the present paper are unstable. But this fact has not precluded physicists from making use of them, just as the demonstrable fact that coplanar rings of electrons in an atom are mathematically impossible, is even now frequently ignored by speculators in atomic theory. In the present case, where not the steady motion of an atom but its stability, is in question, this procedure on the part of physicists is, however, frequently justified. A single ring of electrons in an atom cannot be stable, but the disturbances which break up the atom on account of instability are those which are least likely to occur, so that in spite of this possibility, the atom can have as long an existence as an electrically neutral system, as is demanded by its known properties. Atoms of the type in this Paper, however, do not satisfy this criterion, and their existence could only be very transitory as negatively charged systems. In any other form, or as components of a molecule, they cannot exist at all, apart from all question of stability, and this particular conclusion is not dependent on the system of dynamical reasoning adopted. It appears to vitiate entirely all molecular constructions such as those of Stark.

In the Paper some of the stable vibrations are worked out in special cases for negative systems which would, from their nuclear charge, be identical chemically with others known to exist in the solar corona. Although the full comparison is not given, it may be stated that the coronal spectrum gives no evidence of the existence of such systems. Their instability must be too great. Such a conclusion is in accordance with the very definite instability of their more important modes of vibration.

ABSTRACT.

The Paper is mainly concerned with the possible existence and stability of atoms, and of molecules formed after the manner suggested by Stark, the link between the atoms in a molecule being provided by a stationary electron on the molecular axis. Atoms on the Rutherford model, though dynamically unstable, are stable for the simple vibrations ordinarily excited ; but it is shown in the Paper that atoms with such a stationary electron have a much more limited degree of stability. Moreover, they cannot exist even in an undisturbed state unless they are endowed with a negative charge, for no steady motion is possible, and this conclusion extends even to atoms regulated according to a dynamics such as that of Bohr. Stark's conclusions do not, therefore, survive a quantitative treatment, and molecules cannot be formed in the manner he supposes.

The Paper also discusses the more symmetrical problem, in which there are two such stationary electrons in an undisturbed atom, and it is shown that systems with a transitory existence, which are known by their spectra to occur in the solar corona, are apparently unaccompanied by the still more transitory systems which would be formed by the attachment of an electron after the manner of Stark. This is a further argument against the possibility that two atoms in a molecule can be linked by a single electron, or by two electrons, which attract both atoms.

DISCUSSION.

Mr. T. SMITH asked if it would be possible to make some such generalisation as that symmetry in the system was essential to stability.

Dr. BORNS asked what was meant by the statement that the atom with $+5e$ and $-7e$ is strongly represented in the solar corona.

Dr. R. S. WILLOWS said it was of great advantage that eminent mathematicians were investigating the suggested models of the atom and attempting to compare the results with experiment. Stark had made some brilliant experimental discoveries, but his theoretical deductions appeared to be faulty.

Prof. NICHOLSON, in reply, said he thought some such generalisation as that suggested by Mr. Smith could be made. At any rate, the greater the symmetry of the system the more liable it is to be stable. In reply to Dr. Borns, he meant that certain lines of strong intensity in the spectrum of the corona were referable to such a system.

VI. *Presidential Address, delivered January* 25, 1918, *by* *Prof.* C. V. BOYS, *F.R.S.*

THE cruel exigencies of the war have seriously impeded the work of the Physical Society, as of all our scientific institutions. Many of our members are doing their duty to the country at the front, and many are devoting themselves to problems and hard work imposed by the war conditions, and but little time has been available for those scientific pursuits in which as members of the Physical Society we are normally engaged.

Since my predecessor delivered his address a notable event has occurred which has stirred the scientific community to a degree which I think has been unnecessary. I allude to the passing of the Daylight Saving Bill. Looked at scientifically the thing is ridiculous and a sham, and as such it is naturally hateful to us. But the community at large is not scientific. Thanks to our school education, the general public has a very vague idea of the meaning of time. It has a vague idea that the sun has something to do with the time of day, and that in a crude kind of way it may even be used to indicate the time, but that it is not so good as a clock. In the stress of war the public has been made to realise that it is desirable in the summer months for us to start the day sooner and save, not daylight, but paraffin and gas, and that the ultimate objects for which Mr. Willett strove were in most respects extremely advantageous. The practical and simple operation of putting all the clocks wrong, so avoiding alterations of postal, banking and other business hours, or of reprinting the railway time-tables, though hateful to us in principle, did not disturb the general public, not because all the clocks told the same lie, but because, so far as the multitude were concerned, it was not a lie at all; for, as I have said, they have the vaguest idea of the relation of the sun to time of day, for is it not morning before lunch and after noon after lunch? And they have no understanding at all, either of mean time, of Greenwich or local mean time, or of equation of time. And then when the question was discussed and settled in Parliament we may conclude that the gentlemen there aptly represent their constituents, for the member who introduced the Bill, a member of this Society also, being apparently of that opinion, made fun of the whole system of Greenwich mean time, so that they should not do violence to their collective conscience in passing the Bill.

I do not propose to weary you with a discussion of the metric question, for all that can be said has been said so often, but it is of importance in relation to education. I believe the reason why in England we are at school so backward in mathematics is that the scanty time that is available is for so long not devoted to mathematics at all, but to the wholly unprofitable business of committing to memory tables of weights and measures, and learning all kinds of compound operations which the use of these medieval relics require. While agreeing with other representatives of science that the classical schoolmaster has too much influence in determining the course of our public school education, I do feel very strongly that so long as we can afford to allow our education to be something more than merely utilitarian, we should not give up the study of Latin and Greek, certainly not of both. Besides being the parent of the modern Latin tongues, which seem so well adapted for expressing clearly mathematical and scientific reasoning, Latin gives us a clean cut grammar from which the principles of grammar can be assimilated with a facility that is impossible with a modern language such as English, which has practically lost its grammar. I feel that much of the modern slovenly writing from which authors of scientific Papers are not wholly free, is the result of insufficient literary education. If we had less time more humanly spent we might get further in reading Latin authors, and in addition acquire some facility in writing Latin, and use it later as in the days of Newton as the language of general international communication, at least for our scientific publications.

Another matter of public importance which has relation to the meagre recognition of science as an element of general education has been so consistently urged that I do not care to repeat what we know. I would only say, mainly in answer to any prevalent ideas that our officials need not be scientific, because they can get all the scientific advice that they want, first, that possibly they may not always know when they want —*i.e.*, need—such advice, and, secondly, without scientific training of any kind they may not appreciate the meaning or force of it when they get it. I cannot help being reminded of a statement made by a fellow-student of the School of Mines that in his country there were some people who were so superior to ordinary folk that they would not allow their sons to be taught how to write, as that was clerk's work, and they could get all the writing they wanted done for them without degrading themselves. This may or may not be true, but the parallelism is very close. There is, however, another parallel which I know to be true. When in the early days of the Physical Laboratory in which I spent so much of my activity, an attempt was made to get a lathe, a poor little amateur type of lathe, the attempt was met with an official objection that all the turning that had to be done could be done outside! However, Dr. Guthrie had his way in the end, and that poor, little, despised lathe justified its existence, and I am glad to hear that it is still in use. The change in the official attitude between that time and the present at least as to the requirements of a physical laboratory, may be seen by an examination of the splendid workshop which now exists.

The want of knowledge of the elements of science and hence of proportion and common-sense among those who have to administrate, whether in the higher official ranks in Whitehall, or in more humble local affairs, leads from time to time to fatuous acts or wanton and wicked waste. What better example can we have at a time when fuel is scarce and public waste is more than ever a scandal, than the method of darkening our streets, which after three years still prevails. No doubt there is some reduction in the light actually developed in the street lanterns, but how much reduction there may be no one can tell, for those who are responsible continue to hide their light under a bushel, and produce at least a hundred times as much as they utilise.

I should like to add another example of endless discomfort and expense, not the result of any official ignorance and indifference, but of the want of sense of the community in general, and in many cases of the neglect of the architect to stand between the builder or plumber, whose indifference is so carefully cultivated, and his miserable dupe, the householder. You will not fail to realise the annual agony of the burst water pipe, which results from the builder's habit of getting all pipes to outer walls, and, not content with that, there enclosing them inaccessibly, or to placing service pipes at insufficient depth. The burst water pipe is unknown in really cold countries where the elements of sense are allowed to prevail, and it is unnecessary here. Similarly, during our rare periods of great heat, it is impossible to persuade people in this country to shut the heat out during the day and open to the cool of the night.

As all my contributions to Physical Science have been experimental, and I have consistently devoted myself to the art of experiment, it is possible that some observations on this may be useful. We all have our particular aptitudes, if we have aptitude at all, and it is best for general progress that each individual should, so far as it is possible, be so engaged that his aptitude has play and his activity is in consequence the more successful. In our own science there are widely different ideas prevalent in the different branches. The borderland between Physics and Chemistry, for instance, interests a mind for which the geometrical perfection of Optics or Crystallography might offer no attraction, while it may be almost incomprehensible to those who delight in the mechanical side of physics. Similarly, there are aptitudes even more distinct leading towards abstract, theoretical or mathematical development, or, on the other hand, in the direction of experiment and invention. It is those who feel that this last is the direction that they can follow with most hope of success that in particular I am now addressing. In order to succeed as an experimentalist it is necessary to find by personal experience how as many materials as possible behave under as many conditions as possible, and this can only be done by one who will practise every mechanical art and use every tool and every instrument that he can. In the first place, it will be best to learn in the spirit of the amateur mechanic who wants to make things nearly as well as the professional, and who, therefore, merely imitates the practice of the professional mechanic, but this must not last too long or the would-be experimentalist will not cease to be the amateur mechanic. When each art and the tools and methods employed have been practised so that the operator has some confidence in his judgment as to what will happen, he may and should then use them as required, not as the amateur mechanic does imitating the professionals, and trying with probably less skill to make his work look as much like theirs as possible, but, defying all the traditions of the art, utilise the information so gained to his own ends to help him in his investigation, even though intentionally he does it in a way that would horrify the professional. It is this slavery to tradition and practice that makes the assistance of the professional so tiresome to the experimentalist. So often it takes less time to do without such assistance. I have been constantly impressed by an expression due to Fresnel, quoted to me by a member of this Society: "If you cannot saw with a file or file with a saw, you will be no good as an experimentalist," or words to that effect. If any of our younger members should have any lurking suspicion that understanding and practice of the mechanical arts which is essential to manipulative skill is derogatory, I would urge him to consider what Fresnel did for the science of Optics, and to take his advice to heart. Again, I would urge him to think of what Newton did for Mathematical and Physical Science, and yet such was his experimental aptitude that he made his reflecting telescope (now a cherished possession of the Royal Society) with his own hands, and I believe discovered the use of that most unexpected but necessary substance pitch,* as the material for the polishing tool. Let him think of the helplessness of an experimental chemist who could not perform the elementary operations in glass-blowing, or of the experimental genius of Cavendish or Faraday if he thinks devotion to the art of experiment unnecessary. I know the very natural feeling of distrust in his own ability to construct or to extemporise devices that will really work at all. Even though he may never attain the constructive skill of the professional in any one art, he must practice as required every art, and he should arrive at results as he wants them which meet his requirements better than anything he could get done, because he knows exactly what he wants, and far more quickly as he will, when he has got over his amateur mechanic days, refuse to waste time on conventions of finish or of unnecessary elaboration. Let him beware of listening to the mischievous advice of another whose fingers perhaps are all thumbs, not to waste his precious and valuable time in doing the work of a mechanic, for the experimental art for which he is fitting himself is not the art of a mechanic, but it is the constant use of the knowledge, skill and experience of the mechanic, so far as it can be attained, guided by science and first principles, to produce new or better results than the professional can give him. I do not wish to dwell too much upon any of my own experiences in illustration of this, but I hope I may without offence refer to the development of the radiomicrometer. When I had considered the theoretical possibilities and determined the direction to follow, I found that the success of the instrument should be the greater as the thermo-electric bars could be made smaller in cross-section. The antimony-bismuth alloys so greatly exceeded in thermo-electric power any ductile metals that I knew that I wished to use these, and at the time, having no experience in making delicate thermo-electric couples, I asked the chief designer of one of our leading instrument makers what was the smallest bar that they could make and solder. The cross-section was one or two square millimetres. Now it was essential that I should use bars of not more than about one-twentieth of a square millimetre in section, and I was assured that I could not get that done. It did not take many days to find out how to produce and solder such bars, employing not more than a fifth of a milligramme of solder for each junction, but not by imitating the professional in any respect. I remember that when I was making my early experiments in this class of construction a certain instrument maker called, and I showed him what I wanted to do. He said I had far better let him give it to his really skilled mechanics to do it properly, and not waste my valuable time in vain attempts. Naturally I paid no attention and soon found out how to succeed. And then I was brought up short by the non-existence of any suitable suspending fibre. This led to a further investigation from which the quartz fibre was evolved. First, there was the necessity for making a non-extensible and elastic fibre with a stiffness if necessary one-thousandth or one-ten-thousandth of that of spun glass. Then the trial of materials to find the best. Only glass and the siliceous minerals possessed the property of drawing out into threads, and silica alone when the difficulties of working it were overcome, was found to be the material *par excellence*. Now, I ask the superior person what chance should I have had in arriving at any of these results by going to the professional mechanic. The experimental art, the importance of which I am urging, cannot be done entirely by deputy, nor can it be understood if it is not practised. If any of our members should doubt the truth of what I am saying, he will find innumerable examples of experimental success depending on the experimental art if he will read the published volumes of Lord Rayleigh's researches, in which one of the most inspiring features is the fact that in one research after another existing knowledge is increased by home-made means, which are profoundly delicate or accurate, as the case may be, but are of surpassing simplicity.

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* I am informed by our member, Mr. T. Smith, of the National Physical Laboratory, that Newton probably learned of the use of pitch from Italy.

I have made it my business to use every tool and instrument and to handle every material that I could, and I know no other way of finding out the possibilities of construction. Though written so long ago the famous first three volumes of Holtzappfel are still the best storehouse of information on processes at large, and I take this opportunity of expressing my gratitude for all that I learned from them. I remember on one occasion having the opportunity, not by any means common, of handling five or six large uncut diamonds each as big as a walnut. Now, after getting over the surprise of noticing that diamonds are so heavy, though I knew it as a fact, the first thing to arrest attention was the curious result of bringing two into contact. Of course, everyone who has reached the early stage in manipulation of blowing glass bulbs has noticed the difference in the contact of freshly-blown glass bulbs and of bulbs which have been made any length of time. But the contact of diamonds is unlike either. There is a peculiar slipperiness, due, I think, to the feature I am now about to discuss, which is found to a less degree in a heap of flints. When brought into contact while held lightly in the fingers they emit a curious note, running up into a kind of interrogatory or falsetto squeak. Doing the same with softer materials leads to sounds with most of which we are familiar, but the diamond is unlike any. The soft touch of two corks, or the dead contact of two bullets of lead indicate naturally enough the softness or want of elasticity of these materials, and yet they are each elastic in their way. Harder materials, such as cast brass, and then gunmetal, again produce sounds that differ from the others and from one another, and the sounds give the sensation of increasing hardness or elasticity. Glass and dead-hard steel differ again, but they are not like diamond. There is some sense of bouncing, but it is not till the diamond is used that this squeak is heard. My observation was made years ago, and the opportunity did not last many minutes, so I cannot do more now than give a general impression, though no doubt what was a new experience to me is commonplace enough among those who habitually handle such property. My impression is that the note produced must have run up, perhaps, to 2,000 vibrations per second. Just as a ball under gravity bouncing from the floor makes smaller and smaller excursions at diminishing intervals proportional to the square root of the height, so each diamond lightly held by the soft fingers is under a fairly uniform force directing it towards the other one. I do not remember whether this force was likely to be equal to or be much more than their weight, but, supposing for the moment it to be equal, see what this note means. If the note ever does reach the 2,000 vibrations per second, which is the impression left with any such steady applied force, it means that the diamonds are bouncing when so heard with slowly diminishing excursions of 1/80,000 inch, a phenomenon only possible with a material of such perfect elasticity or hardness. I have been able to turn the information obtained from this observation to account in a curious direction, to which I cannot refer, but I am now mentioning it because it is possible that some light might be thrown on the subject of the determination of hardness of hard materials by tests of this kind. It is true that all that is really observed is the approach to perfection of the coefficient of restitution, but as the superlative hard material has this quality, the question arises whether the two do not go together. Coefficient of restitution or elasticity we do understand in the sense that we can define it exactly and give it dimensions, but hardness is despairingly elusive. We all know the scale of hardness in minerals in which the diamond is pre-eminent. The harder material will scratch the softer, but not be scratched by it. Again, we know that there are certain empirical methods of testing the hardness of metals in the workshop. Of these, the Brinell test, which depends on the measurement of the depression made by a steel ball under pressure is no doubt the best known. There is another test with an instrument called the Shore Scleroscope, which depends on the rebound of a diamond-faced hammer. Another class of test depends on a scratching process, and yet another on resistance to wear. But the results of different tests do not necessarily agree, because though intended to measure hardness they are in reality measuring something different. The subject is discussed very fully in the " Proceedings " of the Mechanical Engineers for November, 1916, a copy of which was kindly sent to me by Sir Robert Hadfield.

The general conclusion come to is that tenacity and elasticity are main factors determining the result of a test. It is also found that when the hardness of dead-hard tool steel is reached the methods of testing have come to their limit. The subject is so curiously involved that it may be well to consider the scratching test. A corner of quartz will scratch hard steel and a sandstone grinder will grind steel. On the other hand, if the scale of pressure is altered the steel tool will plough out a groove in the sandstone. Here no doubt the structure of the hard grains and cementing material is less strong than the material of the grains, and this may be a true explanation so far, but I believe that a properly held steel tool very forcibly applied to a well-supported large crystal of quartz would crush the harder material in front of it, and so could be made to scratch the quartz crystal. If so, what is the meaning of any scratch test ?

At any rate, there is a well-known tool used largely for turning granite and hard rock, in which a hard steel disc dished and narrow at the edge rolls under pressure over the material, which comes away in flakes, while the steel is undamaged. Another factor of the utmost importance is speed, but it would take me too far away to pursue the discussion to this. It is difficult to define hardness. It comes back to a question of tenacity, resistance to crushing or to shearing. This is true of the Brinell test, while the Shore test introduces also elasticity, but it is not only elasticity, for it requires that a permanent indentation be made and work done on the material leading to a reduced rebound. The question arises whether there is any such thing as hardness at all—*i.e.*, in the physical sense—exactly definable and having dimensions. I am almost forced to think not, and that what is appreciated as hardness is a variable and unknown combination of these other qualities. Hardness is the Mrs. Harris of the workshop. This does not prevent the workshop determination of hardness of metals from being of great practical importance, and it is the more complicated by the crystalline structure of metals. For this reason Sir Robert Hadfield has offered a prize for a new and better workshop test of hardness of metal, more especially for the harder kinds, to which I am glad to direct your attention. Now, going back to the diamond and its squeak, it seems possible that a determination under defined conditions of the musical rebound would enable one to differentiate between the hardness of very hard materials from the diamond down to steel or even lower. Where conducting bodies are concerned the use of a Duddell oscillograph would enable one to determine the period of rebound and the rapidity of its diminution, and hence under exactly known pressure conditions the coefficient of restitution. According to the hardness there would be, I expect, an abrupt stoppage of the actual bouncings, so that if for instance, the last bounce of diamond was found to be 1/100,000 inch, sapphire, topaz, quartz and steel would in order show more rapid diminution of period and a larger final excursion. Of course, any such test is unlike tests which depend upon permanent deformation, and being entirely within the elastic limit, it is possibly useless as a workshop test for metals, but it might give some useful information on hard minerals and crystals. In the case of non-conducting crystals the only way of connection with the oscillograph would be through a microphone, for the diminishing period only need be determined ; the diminishing and final terminal amplitude then follow.

Hardness is not the only elusive physical quality which is perplexing engineers at the present time. When the worm-gear designed by Lanchester was tested for efficiency at the National Physical Laboratory in that beautiful testing machine invented and designed by Lanchester also, a machine which directly measures the loss, instead of leaving it to be found as the small difference between two imperfectly known large quantities, it was found that the lubricating property of oil depended on something which at present is unknown. It is not viscosity. Animal and vegetable oils lubricate better—*i.e.*, they are more slippery than mineral oils of the same viscosity—and though the oil trade has known how to make good slippery mixtures no one at present knows what " oiliness " is, and this is at the present time an important physical quest of the engineer. It may, for instance, be as real as hardness, in being a combination quality, and yet share with hardness its relationship with Mrs. Harris.

Here, then, are two very difficult and at present elusive subjects of inquiry, for which the engineer seeks a practical answer, but in so far as any practical answer is based upon exact principle and strict definition will it be likely to approach to a true and complete answer, and this part of the answer is one which Members of this Society might seek for, for it is clearly within the domain of physical science.

While there is always abundant scope for the better utilisation of scientific principles in our manufactures, I would most earnestly caution enthusiasts from our science schools against supposing that manufacturers are ignorant, and that they can easily put them right. I have probably seen more of the development of invention in manufacturing operations in the last quarter of a century than the majority of those concerned in the teaching of science, and I am bound to say that I have been struck on the whole not by the ignorance prevailing, but by the wonderful ingenuity and thorough understanding of what they are doing to be met with in the great factories. Not only, as the scale is increased and special knowledge is impressed, do we find the scientific equipment in men and things the more perfect, but when fiscal conditions allow and the scale is sufficient, the tables are turned, and discovery and research of value to science are made in the factory which would be out of the reach of our scientific institutions or possibly even of our national laboratories, simply on the ground of expense and equipment. The researches connected with the name of Dr. Coolidge, at Schenectady, on tungsten, and the production of targets and the drawing of this refractory metal through red-hot diamonds, appear to me to be an example of such retaliation. The enormous cost of such researches is only possible in the development of a great industrial process, and in such cases pure science is benefited by industry.

In advocating more attention to the cultivation of experimental art, you will understand that I am not suggesting in the slightest degree that theoretical and mathematical study should be neglected. My theme is that experimental art is essential to success, and that it tends to be too much neglected. Just as the classic schoolmaster takes charge in our schools, and makes every other subject give way to his favourite classics, so there is danger of our more theoretical professors failing to realise that there is an immense lore on which the experimental art is based, and that successful experiment is an essential element of scientific progress.

VII. *Recording Thermometer.* *By* C. V. Boys, *F.R.S.*

Received Nov. 15, 1917.

The instrument described in this Paper was designed to meet a special case, and while there is no new scientific principle involved it is so simple and so easily made and its operation is so good that a description of it may be of some interest. The object sought was to obtain a continuous record of the temperature in the case of a regulator clock which I knew to be over-compensated for temperature, but I wished to have something better than readings of a thermometer in the clock at week-ends only, so as to compare the rate as determined by a transit instrument with the mean temperature. I have a fairly good idea of the amount of over-compensation from about 18 months' comparison of rate with the weekly temperature readings, but a continuous temperature record would be much more satisfactory.

Inside a clock-case the pendulum and falling weight break up the somewhat restricted space available, so that an instrument must be adapted to the situation; further it is desirable that the thermometer part of the instrument should extend over about the same limits of level as the gridiron and bob of the pendulum rod. I have, therefore, taken an ebonite rod 35 inches long and $\frac{3}{8}$ inch in diameter and threaded it through a stout glass tube. The rod is sawn through at each end axially for a short distance and cross-drilled so as to take at the lower end a bridge made of a piece of hacksaw blade and at the upper end the short arm of a lever of which the fulcrum is also a piece of hacksaw blade. A notch is filed diametrically across the glass tube at its lower end for the bridge to bear against, while a corresponding notch is filed across the glass tube at the other end about tangent to its inner surface. A brass ring is cemented to the outside of the tube a few inches from its upper end, and a spring (Fig. 2) connecting the long arm of the lever with this ring keeps both knife edges in firm contact with the file notches. This part then is a self-contained whole and the end of the long arm moves about 7 times as much as the ebonite within the tube. The tube is suspended in the corner of the clock out of reach of the pendulum from a bracket, which carries also a cantilever support adjustable in all directions for the axis of the recording pen. This also is provided with a short arm which comes below the end of the long arm of the thermometer, and a long arm reaching across the clock between the pendulum gridiron and the weight to the pen which makes its record on a drum on the other side. The weight of the long arm of the pen causes the short arm to rise as far as it can, but it is controlled by a steel rod pointed at each end. The points rest in centre punch dots above the short arm of the pen lever and below the long arm of the thermometer lever. There are a number of these dots and it is a small matter to shift the rod from one to another until the desired scale of magnification is reached. A piece of cotton is tied to the steel rod and to the cantilever, so that if the rod is dropped it does not descend into the depths. I have found the dimensions stated below give a movement of 1 inch for 10°F. at temperatures of 40°–55°F. Having in the country no means other than the seasons for regulating the temperature, I have not at present determined to what extent the scale is more open at higher temperatures. Ultimately I should rule paper to agree with the instrument rather than try by inclination or slight bell-cranking of the levers to cause them to introduce an approximate correction. I have made the pen of silver, somewhat like a drawing pen of a box of instruments but without a screw and very small and well finished, and it rules a very fine clean line. The record can be read to $\frac{1}{20}$° F., and the whole arrangement is so rigid that it could certainly be used with more magnification to record to $\frac{1}{100}$° C. on an empirical scale, which would have to be determined.

It seemed absurd to put a driving clock into a drum already situated in a clock, and so I have made the following arrangement for moving the drum. The drum is a free fit upon a wheel with tubular centre which turns on an upright steel rod. The drum may thus be lifted off to change the paper. The wheel rests upon a washer and boss. Under the horizontal wheel a radial axis is placed with its inner pivot close to the boss and its outer end resting by means of a groove in an upturned *V* on an elastic arm. Beyond the groove and overhanging a 4-inch pulley of aluminium is fixed to the radial axis. Between the groove and the inner end a roller of ebonite about 0·3 inch in diameter is also fixed, and this is pressed by the elasticity of the arm and *V* up against the lower surface of the wheel. The two surfaces are smooth and conical, but the ebonite cone is slightly convex so that it may be adjusted and still give approximately correct rolling. A cord from the bottom of the clock weight passes under a pulley on a weight let down so as to rest on the floor of the clock under the driving weight up over the aluminium pulley to a peculiar weight to be described later. As the clock-weight falls $1\frac{1}{2}$ inches a day the drum rotates $\frac{1}{8}$ inch a day, and so the ordinary barometer charts are available, the two-hour periods becoming days and the horizontal $\frac{1}{20}$ inch spaces being $\frac{1}{2}$° F. In the place of 28, 29, 30, 31 inches are written 40°, 50°, 60°, 70°, but when the winter had fairly started these were made 30°, 40°, 50°, 60°, and in the summer 50°, 60°, 70°, 80° may be chosen. There is a screw adjustment under the pen carrier, shown in Fig. 2 so that it may be adjusted in a moment so as to read correctly.

I have said that the small weight is peculiar. It is desirable when the clock is wound, which need be only once a month, that the drum should not unwind and require resetting, and it is desirable that the small weight with its smaller fall should not limit the time that the clock may be left unattended. Further it is desirable that the pull on the cord below the weight should not appreciably alter the driving force of the weight. These three ends are attained by taking the cord round a pulley on the small weight and up to a nail which halves the fall, and then by fixing two pieces of wine cork very nearly in contact above the pulley and leading the cord round these two quarter turns, so that it enters and leaves adjacent to itself. Owing to the friction of nearly half a turn of cork the entering cord and the leaving cord are very differently loaded, the ratio $e^{\mu\theta}$ being perhaps 1 to 5. Taking this as an example the cord round the pulley carries only one-sixth of the small weight when the clock is going, and this is all that is added to the pull of the clock weight, while the nail takes five-sixths. The aluminium pulley is therefore pulled down as by $\frac{1}{3}$ of the small weight and the spring *V* support overcomes this and keeps the ebonite roller in driving contact with the drum. When, however, the clock is wound, the cord over the aluminium pulley, which had been the entering cord, becomes the leaving cord and its tension is increased five fold, and the aluminium pulley now carrying $1\frac{2}{3}$ of the small weight overcomes the spring *V* support and the ebonite roller moves down out of contact with the wheel carrying the drum, through about a thousandth of an inch, where it is limited by a screw stop. A second screw is provided to boost the spring if necessary (Fig. 3). It would not do simply to pass the cord round a pulley on the small weight secured thereto so that it could not turn, because with the two cords so far apart a considerable movement would occur before the transfer of the load by the rocking of the weight was sufficient to put the spring *V* into action.

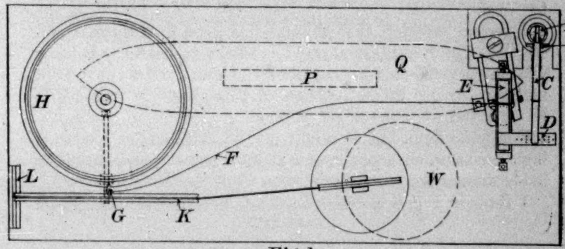

Fig. 1

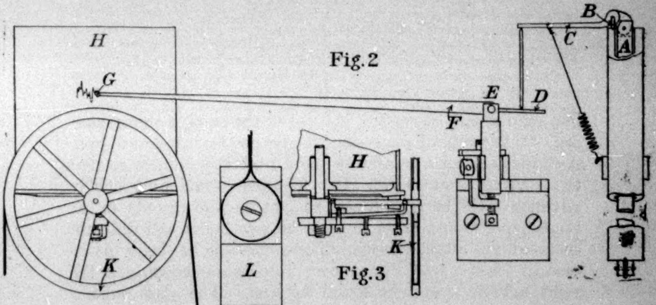

Fig. 2

Fig. 3

Fig. 4

A, the ebonite rod in its glass tube.
B, upper knife edge.
C, long arm of thermometer lever, 2 inches (short arm about 0·3 inch).
D, short arm of pen lever 0·7 inch.
E, axis of pen lever.
F, long arm of pen lever 8½ inches.
G, pen.
H, recording drum.
K, aluminium wheel.
L, small weight.
P, space occupied by pendulum rod.
Q, space at lower level occupied by pendulum bob.
W, space occupied by driving weight.

Fig. 1 is a plan showing the position of the parts of the instrument as placed in the clock. Fig. 2 is an elevation in which the thermometer is removed from its bracket and turned round about the pointed steel rod as an axis so as to show more clearly the operation. The insets Figs. 3 and 4 show the small weight and the spring *V*, &c., used for driving the drum.

I should mention that ebonite was used by Edison about 40 years ago in the instrument he called a tasimeter. It has a very large coefficient of expansion which makes it a convenient material for use in instruments of this class. It would also be an extremely convenient expanding element in a compensation pendulum, but it might not be suitable for clocks of high precision on account of a slight want of proportionality in its expansion.

I show in Fig. 5 a design for a temperature recorder to go into an ordinary barograph in front of the barograph mechanism. It seems hardly necessary to give any further description. A convenient scale to aim at would be 1°F.$=\frac{1}{20}$ inch, so that the 3 inches would cover 60°. The pen would be $\frac{1}{8}$ inch in advance of the barograph pen, so that the two records on the same paper would differ in time by two hours and different coloured inks might be used. The connecting bar being in this case in tension would hook over notches as shown, and the right notches would be found by trial. The glass tube is shown cemented into a collar near one end resting on trunnion pivots, and it can be raised or lowered by a screw lever at the other end to adjust the position of the pen on the scale. A spring band to the right of the collar draws the long arm of the thermometer lever towards the glass tube, thus keeping the two knife edges in firm contact with their *V*'s. With a length of ebonite of 6 inches a magnification of 220 would be needed, which would conveniently be obtained with a first lever magnification of 20 and a pen lever magnification of 11.

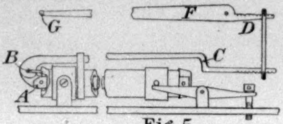

Fig. 5

It will be noticed that with this design the magnification increases with the square of the length.

ABSTRACT.

This instrument was designed and constructed to go into the case of a regulator clock. The thermometric element consists of a rod of ebonite within a glass tube. The differential expansion is determined by a pair of levers giving a movement of 1 in. for 10°F. The drum carries an ordinary barometer chart, and is driven at such a speed that a two-hour interval of $\frac{1}{8}$ in. is passed in 24 hours. The drum is driven by friction by means of a cord from below the driving weight of the clock by an $e^{\mu\theta}$ arrangement, in virtue of which when the clockweight descends the drum turns, but when the clock is wound the drum remains at rest. The instrument is designed with a view to easy construction and accuracy. It is extremely rigid, and much more magnification might be used.

An alternative design on the same lines to go into a recording baragraph is also given.

DISCUSSION.

Mr. C. C. Paterson said that for certain purposes a quick-acting sensitive recording thermometer was very useful. What was the time lag of this instrument in taking up the temperature of the surroundings?

Prof. Lees asked how the ebonite behaved as regards constancy.

Prof. Boys, in reply, said that for his purpose the thermometer was desired to be slow and sluggish, so as not to take up every trifling variation of temperature due to people entering the room. To make it quick acting, a thin strip of ebonite would have to be used, and the glass tube should be replaced by a stout glass rod or plate some distance from the ebonite. As regards constancy, he could only speak for the three months that the instrument had been in use. There had been no signs of variability in that time.

VIII. *The Primary Monochromatic Aberrations of a Centred Optical System.* *By* S. D. Chalmers, *M.A., Technical Optics Department of the Northampton Institute.*

Received November 14, 1917.

In considering the defects of an optical system the first step is to obtain a simple theory of optical systems, ignoring all those effects which tend to produce bad definition or distortion of the image. This theory should furnish the ideal positions and sizes of the images to serve as comparisons with the actual results obtained by the system.

The next step is to take into account the most important of the causes which contribute to the defects of simple lenses when they are used with moderate apertures and fields. These defects will be referred to as Aberrations of the First Order.

In practical systems these defects are usually compensated by other lenses, and the residual defects are made up of the uncompensated effects of the First Order and other defects which are ignored in this treatment, and will be referred to as Aberrations of Higher Order.

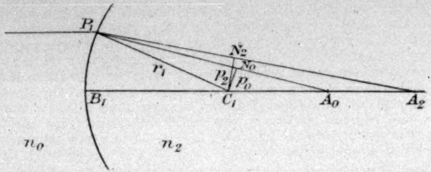

Fig. 1.

In the treatment of the First Order Aberrations two methods will be used, one primarily suited to the case where the separation of the surfaces is small, and the other more suitable to the case where the separations vitally affect the design.

The aberrational defects are expressed as lateral aberrations, *i.e.*, as defects measured in the focal plane of the system. The procedure adopted is to express the aberrational defect of a single surface in terms of the constants of the surface and the perpendicular distance of the ray considered from the centre of curvature of the surface.

The value of this perpendicular can be expressed in terms of the co-ordinates of the ray in any chosen medium, and thus the aberration due to each surface can be expressed in terms of the co-ordinates of the chosen ray, in such a way that the aberrations of the individual surfaces can be summed.

Refraction at a Single Surface.

Let P_1B_1 (Fig. 1) be a spherical surface of radius r_1 separating the two media n_0 and n_2. Let C_1 be the centre of curvature, B_1 the vertex and P_1 any point on the surface, and let the ray which was travelling in the direction P_1A_0 in medium n_0 be deviated to P_1A_2 in medium n_2.

Denoting the angles $C_1P_1A_0$ by i_1 and $C_1P_1A_2$ by j_1 the law of refraction gives

$$n_0 \sin i_1 = n_2 \sin j_1.$$

In the triangle $C_1P_1A_0$, we have—

$$\frac{\sin C_1}{\sin i_1} = \frac{P_1A_0}{C_1A_0},$$

so that—

$$\frac{P_1A_0}{n_0C_1A_0} = \frac{\sin C_1}{n_0 \sin i_1} = \frac{\sin C_1}{n_2 \sin j_1} = \frac{P_1A_2}{n_2C_1A_2}. \quad . \quad . \quad (1)$$

If the perpendiculars $C_1N_0 \equiv p_0$ and $C_1N_2 \equiv p_2$ be dropped from C_1 on P_1A_0 and P_1A_2 respectively,

$$P_1A_0 = (C_1P^2 - p_0^2)^{\frac{1}{2}} + (C_1A_0^2 - p_0^2)^{\frac{1}{2}},$$

and this expression can be expanded in powers of p_0^2,

$$P_1A_0 = C_1P + C_1A_0 - \tfrac{1}{2}p_0^2\left(\frac{1}{C_1P_1} + \frac{1}{C_1A_0}\right) + \text{terms in } p_0^4, \text{ \&c.}$$

Similarly,

$$P_1A_2 = C_1P + C_1A_2 - \tfrac{1}{2}p_2^2\left(\frac{1}{C_1P_1} + \frac{1}{C_1A_2}\right) + \ . \ . \ . p_2^4, \text{ \&c.}$$

Writing $C_1P = r_1$ we have from equation (1)

$$\frac{r_1 + C_1A_0}{n_0C_1A_0} - \frac{1}{2}\frac{p_0^2}{n_0C_1A_0}\left(\frac{1}{r_1} + \frac{1}{C_1A_0}\right) + \ . \ . \ .$$

$$= \frac{r_1 + C_1A_2}{n_2C_1A_2} - \frac{1}{2}\frac{p_2^2}{n_2C_1A_2}\left(\frac{1}{r_1} + \frac{1}{C_1A_2}\right) + \ . \ . \ . \ . \ . \quad (2)$$

and we shall approximate at first by ignoring all terms in p_0^2 and p_2^2, but shall take these terms into account later on. If the terms in p_0^2 and p_2^2 can be ignored, all rays proceeding to A_0 will be deviated to A_2, whatever be the position of P_1 and A_2 will be the image of A_0.

To this approximation

$$\frac{r_1+C_1A_0}{n_0C_1A_0}=\frac{r_1+C_1A_2}{n_2C_1A_2}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Dividing by r_1 this can be expressed as

$$\frac{1}{n_0C_1A_0}+\frac{1}{n_0r_1}=\frac{1}{n_2C_1A_2}+\frac{1}{n_2r_1}. \quad . \quad . \quad . \qquad (4)$$

If a small object A_0E_0 at right angles to the axis be considered, and a plane be drawn through A_2 at right angles to the axis, a ray proceeding to E_0 and passing through C_0 will not be deviated; it will therefore meet the plane through A_2 in E_2, such that

$$\frac{A_2E_2}{A_0E_0}=\frac{C_1A_2}{C_1A_0}.$$

The point E_2 will be a point in the image patch of E_0, and may be regarded as the ideal image of E_0. In this case the object A_0E_0 is reproduced of the size A_2E_2, and $\frac{A_2E_2}{A_0E_0}$ is termed the linear magnification, and will be denoted by m_1.

For convenience we use the value $\frac{n_2A_2E_2}{n_0A_0E_0}$, and denote this by M_1.

The quantities $\frac{1}{n_0C_1A_0}$ and $\frac{1}{n_2C_1A_2}$ are denoted by S_1 and T_1 respectively, and the relation (4) becomes $T_1-S_1=\left(\frac{1}{n_0}-\frac{1}{n_2}\right)\frac{1}{r_1}$, and the last term being independent of A_2 and A_0 is a property of the surface only, and may be denoted by P_1.

We thus have

$$T_1-S_1=P_1$$

and

$$M_1=S_1/T_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

From these two relations we deduce

$$M_1=1-\frac{P_1}{T_1},$$

$$\frac{1}{M_1}=1+\frac{P_1}{S_1}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

or, writing $t_1=\frac{1}{T_1}$ and $s_1=\frac{1}{S_1}$

$$M_1=1-t_1P_1 \frac{1}{M_1}=1+s_1P_1.$$

These formulæ can be applied to any system of lenses; let $P_{1\cdot3}=T_1/M_3$, $n_2C_1C_3=a_2$, &c., and let $M_3s_3t_3P_3S_3$ and T_3 relate to the next surface.

Then, since $\frac{1}{M_3}=1+\frac{P_3}{S_3}$,

$$T_{1\cdot3}=(1+s_3P_3)T_1=T_1+\frac{t_1-a_2}{t_1}P_3.$$

If a ray be drawn through the system from the object point on the axis, so that the perpendicular distance from $C_1=p_0$ then the optical length of this perpendicular would be n_0p_0, and this would equal the corresponding quantity after refraction; denoting this quantity by q_1 and the corresponding quantities for the other surfaces by q_3, we have

$$\frac{q_3}{q_1}=\frac{s_3}{t_1}=1-a_2T_1,$$

so that the quantity $1-a_2T_1$ represents the optical length of the perpendicular from C_3, as compared with the optical length of the perpendicular from C_1.

The method can be extended to more surfaces, and we have

$$T=T_1+\frac{q_3}{q_1}P_3+\frac{q_5}{q_1}P_5, \quad \ldots \ldots \quad (7)$$

and

$$\frac{q_{2r-1}}{q_1}=\frac{q_{2r-3}}{q_1}=a_{2r-2}T_{1,2r-3}. \quad \ldots \ldots \quad (8)$$

Also

$$t_{2r-1}=\frac{q_{2r-1}}{q_1}/T. \quad \ldots \ldots \quad {}^*(9)$$

If $S_1=O$, $T_1=P_1$, and T is analogous to the equivalent focal power of the system, denoting this value of T by P, we have

$$M=\frac{q_{2r-1}}{q_1}-t_{2r-1}P, \quad \ldots \ldots \quad (10a)$$

as the relation between M, P and T_{2r-1} for the complete system, where $\frac{q_{2r-1}}{q_1}$ and P are calculated for $S_1=o$, but M and T_{2r-1} relate to any object point.

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* Note.—p_1 would then relate to any ray, while q_1 relates only to this selected ray drawn from an object point on the axis.

The analogous relation between s_1 and M is obtained by considering the reversed system, and is

$$\frac{1}{M}=\frac{Q_1}{Q}+s_1P, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (10b)$$

where Q_1 and Q denote the optical lengths of the perpendiculars from C_1, and the centre of curvature of the last surface, on a ray incident parallel to the axis in the last medium.

Aberrations of a Single Surface.

Returning to equation (2) we have to the next approximation

$$\frac{r_1+C_1A_2}{n_2C_1A_2}=\frac{r_1+C_1A_0}{n_0C_1A_0}+\frac{1}{2}\frac{p_2^2}{n_2C_1A_2}\left(\frac{1}{r_1}+\frac{1}{C_1A_2}\right)-\frac{1}{2}\frac{p_0^2}{n_0C_1A_0}\left(\frac{1}{r_1}+\frac{1}{C_1A_0}\right),$$

on dividing by r_1

$$\frac{1}{n_2C_1A_2}+\frac{1}{n_2r_1}-\left(\frac{1}{n_0C_1A_0}+\frac{1}{n_0r_1}\right)=+\frac{1}{2}\frac{p_2^2}{n_2\,.\,C_1A_2\,.\,r_1}\left(\frac{1}{r_1}+\frac{1}{C_1A_2}\right)$$

$$-\frac{1}{2}\frac{p_0^2}{n_0\,.\,C_1A_0\,.\,r_1}\left(\frac{1}{r_1}+\frac{1}{C_1A_0}\right), \qquad (11)$$

but $n_2p_2=n_0p_0$, and this may be denoted by p_1.

Also $$\frac{1}{n_2C_1A_2}+\frac{1}{n_2r_1}=\frac{1}{n_0C_1A_0}+\frac{1}{n_0r_1}$$

to the first approximation.

Substituting this in the terms of higher order the value of $\frac{1}{n_2C_1A_2}$ exceeds its ideal value by

$$\delta\frac{1}{n_2C_1A_2}=+\frac{1}{2}\frac{p_1^2}{r_1}\cdot\left(\frac{1}{n_2^2C_1A_2}-\frac{1}{n_0^2}\frac{1}{C_1A_0}\right)\left(\frac{1}{r_1}+\frac{1}{C_1A_2}\right)\frac{1}{n_2}.$$

And

$$\delta\frac{1}{n_2C_1A_2}=-\frac{1}{n_2C_1A_2^2}\cdot\delta C_1A_2$$

$$=\frac{1}{2}\frac{p_1^2}{r_1}\left(\frac{1}{n_2^2C_1A_1}-\frac{1}{n_0^2}\frac{1}{C_1A_0}\right)\left(\frac{1}{r_1}+\frac{1}{C_1A_2}\right)\frac{1}{n_2}.$$

But the lateral aberration is parallel to p_2 and $=\frac{p_2}{C_1A_2}.\delta C_1A_2$, so that the lateral aberration is

$$\frac{p_1}{n_2C_1A_2}\times -n_2C_1A_2{}^2\cdot\tfrac{1}{2}\frac{p_1{}^2}{r_1}\left(\frac{1}{n_2{}^2C_1A_2}-\frac{1}{n_0{}^2}\frac{1}{C_1A_0}\right)\left(\frac{1}{n_2r_1}+\frac{1}{n_2C_1A_2}\right)$$

$$=p_1\left(-\tfrac{1}{2}p_1{}^2\frac{C_1A_2}{r_1}\right)\left(\frac{1}{n_2{}^2C_1A_2}-\frac{1}{n_0{}^2}\frac{1}{C_1A_0}\right)\left(\frac{1}{n_2r_1}+\frac{1}{n_2C_1A_2}\right).$$

This is measured parallel to p_2 and its co-ordinates are obtained by substituting the co-ordinates of p_1 in the first term of this expression.

Thus denoting these co-ordinates of the aberration by $[\Delta]$ we have

$$[\Delta]=-\tfrac{1}{2}[p_1]\frac{C_1A_2}{r_1}{}^2p_1{}^2\left\{\frac{1}{n_2{}^2C_1A_2}-\frac{1}{n_0{}^2}\frac{1}{C_1A_0}\right\}\left\{\frac{1}{n_2r_1}+\frac{1}{n_2C_1A_2}\right\} \tag{12}$$

Rays from a Point Not on the Axis.

This expression will also give the aberration of an oblique ray, relative to the ray which passes through the centre of curvature, provided this aberration be measured at the sphere of radius C_1A_2. But it is usually more convenient to express this difference on the focal plane, and there arise additional terms which must be included.

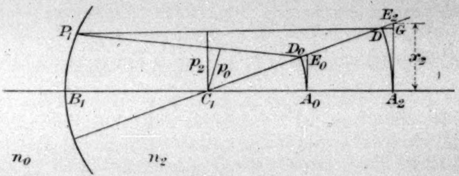

FIG. 2.

If we represent the distances of the image and object from the axis by a_0 and x_2 we have from Fig. 2.

$$EG=\frac{-p_2}{C_1A_2}\times ED=\frac{-p_2}{C_1A_2}\times\frac{x_2{}^2}{2C_1A_2},$$

but there is a similar term

$$\frac{-p_0}{C_1A_0}\;\frac{x_0{}^2}{2C_1A_0},$$

which affects the ray before refraction, and this must be multiplied by $\frac{x_2}{x_0}$, the magnification produced by the surface, and $\frac{x_2}{x_0}=\frac{C_1A_2}{C_1A_0}$ to the first approximation.

Thus, the additional term is

$$\frac{x_2^2}{2C_1A_2}\left\{\frac{-p_2n_2}{C_1A_2n_2}+\frac{p_0n_0}{n_0C_1A_0}\right\}=\frac{-p_2n_2}{2C_1A_2}x_2^2\left\{\frac{1}{n_2C_1A_2}-\frac{1}{n_0C_1A_0}\right\}, \quad (13)$$

and once more the co-ordinates of p_2 can be substituted for p_2 to give the co-ordinates of the lateral aberration. Thus, the complete lateral aberration arising from any surface—say, (5)—and measured on the image plane in medium (6) is

$$[\Delta_6]=-\tfrac{1}{2}[p_5]\left\{\frac{C_5A_6}{r_5}(p_5)^2\left(\frac{1}{n_6^2C_5A_6}-\frac{1}{n_4^2C_5A_4}\right)\left(\frac{1}{n_6C_5A_6}+\frac{1}{n_6r_5}\right)\right.$$
$$\left.+\frac{x_6^2}{C_5A_6}\left(\frac{1}{n_6C_5A_6}-\frac{1}{n_4C_5A_4}\right)\right\}. \quad (14)$$

This can be expressed in terms of s and t as follows :—

$$[\Delta_6']\equiv[n_6\Delta_6]=-\tfrac{1}{2}[p_5]\left\{\frac{t_5}{r_5}p_5^2\left(\frac{T_5}{n_6}-\frac{S_5}{n_4}\right)\left(T_5+\frac{R_5}{n_6}\right)\right.$$
$$\left.+(n_6x_6)^2T_5(T_5-S_5)\right\}, \quad . \; . \; . \; . \; . \quad (15)$$

and

$$T_5+\frac{R_5}{n_6}=S_5+\frac{R_5}{n_4}=T_5+\frac{1}{n_6}\,\frac{T_5-S_5}{\frac{1}{n_4}-\frac{1}{n_6}}=\frac{T_5\frac{1}{n_4}-S_5\frac{1}{n_6}}{\frac{1}{n_4}-\frac{1}{n_6}}$$
$$=\frac{n_6T_5-n_4S_5}{n_6-n_4},$$

also $$T_5-S_5=P_5.$$

If we desire to express $[\Delta_6']$ in the last medium we must multiply by each subsequent M, *i.e.*, by $T_{1\cdot3\cdot5}/T$, or n_6x_6 by $\frac{nx}{n_6x_6}$.

We have $$T_5=T_{1\cdot3\cdot5}\frac{q_1}{q_5} \quad S_5=T_{1\cdot3}\frac{q_1}{q_5},$$

so that $[\Delta']_6$ the co-ordinates in the last medium of the aberration due to surface 5 are

$$[\Delta']_6 = -\tfrac{1}{2}\left[\frac{p_5 q_1}{q_5}\right]\left\{\left(\frac{p_5 q_1}{q_5}\right)^2 \frac{1}{r_5}\left(\frac{T_{1\cdot3\cdot5}}{n_6} - \frac{T_{1\cdot3}}{n_4}\right)\left(\frac{n_6 T_{1\cdot3\cdot5} - n_4 T_{1\cdot3}}{n_6 - n_4}\right) \frac{1}{T}\left(\frac{q_5}{q_1}\right)^2 + (nx)^2 T_5\right\} \quad . \quad . \quad . \quad . \quad (16)$$

The coefficient

$$\frac{1}{r_5}\left(\frac{T_{1\cdot3\cdot5}}{n_6} - \frac{T_{1\cdot3}}{n_4}\right)\left(\frac{T_{1\cdot3\cdot5} n_6 - T_{1\cdot3} n_4}{n_6 - n_4}\right)\frac{1}{T}$$

is in a form suitable for calculation, but it is sometimes more convenient to express it as

$$\frac{R_5 . n_6 - n_4}{(n_6 - n_4)^2}\left(T_{1\cdot3\cdot5}^2 - T_{1\cdot3} T_{1\cdot3\cdot5}\left(\frac{n_6}{n_4} + \frac{n_4}{n_6}\right) + T_{1\cdot3}^2\right)\frac{1}{T}$$

$$= \frac{1}{T}\,\frac{P_5 \,.\, n_4 n_6}{(n_6 - n_4)^2}\left\{(T_{1\cdot3\cdot5} - T_{1\cdot3})^2 - T_{1\cdot3} T_{1\cdot3\cdot5}\,\frac{n_4^2 - 2 n_4 n_6 + n_6^2}{n_4 n_6}\right\}$$

$$= \frac{1}{T}\,\frac{P_5 . n_4 n_6}{(n_6 - n_4)^2}\left\{\left(\frac{q_5 P_5}{q_1}\right)^2 - T_{1\cdot3} T_{1\cdot3\cdot5}\,\frac{(n_4 - n_6)^2}{n_4 n_6}\right\}$$

$$= \frac{1}{T}\quad P_5\left\{\left(\frac{q_5 P_5}{q_1}\right)^2 \frac{n_4 n_6}{(n_6 - n_4)^2} - T_{1\cdot3} T_{1\cdot3\cdot5}\right\} \quad . \quad . \quad . \quad . \quad (17)$$

It now remains to express $p_5 \dfrac{q_1}{q_5}$ in terms of the quantities which can be readily calculated for an individual ray.

Using optical lengths throughout let x_0 be any object point, $x_1 y_1$ the co-ordinates at which it passes the plane (1).

Let p_5' be the image of p_5 in the medium O and let its distance from the plane (1) be b_5, then, from Fig. 3 the co-ordinates of p_5' are

$$[p_5'] = x_0 \frac{b_5}{s} + x_1 \frac{s - b_5}{s},\; y_1 \frac{s - b_5}{s},$$

where s is the distance from plane (1) to the object.

If the intercepts of the ray from which q_1, q_3, q_5 are calculated on the image planes of 1, 3, 5, &c., in medium O be, we have

$$\frac{p_5 q_1}{q_5} = \frac{p_5' q_1}{q_5} \quad \text{and} \quad \frac{q_1}{q_5'} = \frac{s}{s - b_5},$$

so that—

$$\left[\frac{p_5 q_1}{q_5}\right] = x_1 \frac{s - b_5}{s}\,\frac{q_1}{q_5'} + x_0 \frac{b_5}{q_5'}\,\frac{q_1}{s},\; y_1 \frac{s - b_5}{s}\,\frac{q_1}{q_5'} = x_1 + \frac{x}{s}\,\frac{b_5 q_1}{q_5'},\; y_1. \quad (18)$$

If we denote the co-ordinates $x_1 y_1$ by $\rho \cos\psi$ and $\rho \sin\psi$ and let the ray from $x_0 = O$, through $x_1 y_1$ make the angle A, with the axis in the final medium, then

$$x_1 T = A \cos\psi \quad \text{and} \quad y_1 T = A \sin\psi,$$

where the approximations do not determine whether the sine, circular measure or tan of A is to be taken.

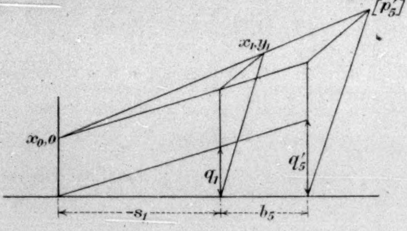

FIG. 3.

Writing $\dfrac{b_5 q_1}{q_5'} T = B_5$, we have

$$[\Delta']_6 = -\tfrac{1}{2}\frac{1}{P}[A \cos\psi + xTB_5 A \sin\psi]\Big\{(A^2 + 2AxT \cos\psi B_5 + (xT)^2 B_5{}^2)\Big(\Big(\frac{q_5}{q_1}\Big)^2 \frac{P_5}{P}\Big(\Big(\frac{q_5}{q_1}\Big)\frac{P_5}{P}\Big)^2 \frac{n_4 n_6}{(n_6 - n_4)^2} - \Big(\frac{T_{1\cdot3\cdot5} T_{1\cdot3}}{T^2}\Big)\Big) + \frac{P_5}{T}(xT)^2\Big\}. \quad . \quad . \quad (19)$$

Thus, writing E_5 for the expression

$$\frac{q_5}{q_1}\,\frac{q_5 P_5}{q_1 T}\Big(\Big(\frac{q_5}{q_1}\,\frac{P_5}{T}\Big)^2\Big(\frac{n_4 n_6}{(n_6 - n_4)^2}\Big) - \frac{T_{1\cdot3\cdot5} T_{1\cdot3}}{T^2}\Big),$$

we have the two co-ordinates

$$(\delta n x)_6 = -\tfrac{1}{2}\frac{1}{T}\Big\{E_5 A^3 \cos\psi + E_5 B_5 A^2 xT(2\cos{}^2\psi + 1) + 3E_5 B_5{}^2 A \cos\psi (xT)^2 + EB_5{}^3 (xT)^3 + \frac{P_5}{T} A \cos\psi (xT)^2 + B_5 \frac{P_5}{T}(xT)^3\Big\}$$

$$(\delta n y)_6 = -\tfrac{1}{2}\frac{1}{T}\Big\{E_5 A^3 \sin\psi + E_5 B_5 A^2 xT(2 \sin\psi \cos\psi) + E_5 B_5{}^2 A \sin\psi (xT)^2 + \frac{P_5}{T} A \sin\psi (xP)^2\Big\}. \quad (20)$$

The quantity $B_5=\frac{b_5 q_1}{q_5'}T$ can be expressed in two forms.

The quantity $\frac{b_5 q_1}{q_5'}$ is the sum of quantities of the form

$$\frac{a_4}{\frac{q_3}{q_1}\frac{q_5}{q_1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

taken from the plane (1) to the central plane considered.

This is a special case of the general theorem, that the apparent distance of any point on an optical system may be obtained as follows :—

The actual distance from any plane in the first medium is divided into parts by the various central planes, each part is multiplied by the refractive index of the medium and divided by the product of the values of q (calculated from any ray) at the bounding planes and the sum gives the corresponding quantity for the apparent distance in the first medium.

A demonstration of the analogous case where the distances are subdivided by the surfaces themselves is given in a Paper by the author.*

Secondly, let the ray parallel to the axis at the height Q in the last medium meet the various central planes in intercepts (of optical length) $Q_1 Q_3 Q_5 \ldots Q$, and let the corresponding intercepts on the image planes in the first medium be $Q_1' Q_3' Q_5' Q'$. Then

$$\frac{Q_5'}{Q_5}=\frac{q_5'}{q_5}.$$

$\frac{Q_5'-Q_1'}{Q}=b_5 P$, where P is calculated for this ray.

The relation between P' and P is given by

$$\frac{T}{S_1}=1/m=\frac{Q_1}{Q}+\frac{P}{S_1},$$

see equation (10*b*), so that

$$P=T-\frac{Q_1}{Q}S_1.$$

Also $$\frac{q_5'-q_1}{q_1}=-b_5 S_1$$

from Fig. 3, so that

$$\frac{Q_5'}{Q_1}-\frac{q_5'}{q_1}=b_5 P\frac{Q}{Q_1}+b_5 S_1=b_5 T\frac{Q}{Q_1},$$

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* Trans. Opt. Soc., Vol. XVIII., April, 1917.

or
$$\left(\frac{Q_5}{Q_1}-\frac{q_5}{q_1}\right)\frac{q_5'}{q_1}=b_5T\frac{Q}{Q_1},$$

i.e.,
$$\left(\frac{Q_5}{Q_1}-\frac{q_5}{q_1}\right)\frac{q_1}{q_5}=\frac{b_5q_1}{q_5}T\frac{Q}{Q_1},$$

i.e.,
$$\frac{b_5q_1}{q_5'}T=\left(\frac{Q_5}{Q_1}\bigg/\frac{q_5}{q_1}-\frac{Q_1}{Q}\right).$$

Thus, B_5T may be expressed in either of the two forms

$$\sum\frac{a_4}{\frac{q_3}{q_1}\cdot\frac{q_5}{q}}P \text{ or } \left(\frac{Q_5}{Q}\bigg/\frac{q_5}{q_1}-\frac{Q_1}{Q}\right) \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where q_1 . . . q_5 are calculated for the actual object point, and Q_1 . . . Q_5Q for a ray incident parallel to the axis in the last medium.

In the above it has been assumed that the ray is best specified by its points of crossing the first central plane and the object plane, but in practice it is more convenient to choose the stop plane for the co-ordinates x_1y_1, and in this case we use the same method, except that surface 1 is taken to be a surface of zero power—*i.e.*, no change of refractive index—with its central surface on the stop plane. The analysis is then unaltered.

In actual practice there is little difference in the amount of computation to obtain B_5T from the two expressions given above.

Thus, the resultant aberrational defects may be obtained by summing the effects due to all surfaces and we obtain the five aberrational coefficients

$$\begin{aligned}E&=\Sigma E_6\\ I&=\Sigma E_6B_6\\ K&=\Sigma E_6B_6^2\\ G&=K+\Sigma P_5/T\\ H&=\Sigma GB_6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)\end{aligned}$$

A sample calculation is annexed.

Interpretation of Results.

I. For the terms independent of θ we have

$$\delta nx=-\tfrac{1}{2}\frac{1}{T}EA^3\cos\psi,$$

$$\delta ny=-\tfrac{1}{2}\frac{1}{T}EA^3\sin\psi,$$

so that each circle on the stop plane corresponds to a small circle in the image and the radius varies as A^3. This defect is uniform over the field, and is referred to as Spherical Aberration of the First Order.

II. The coefficient I. gives rise to terms

$$\delta nx = -\tfrac{1}{2}\frac{1}{T} IA^2 \tan\theta\ (\cos 2\psi + 1),$$

$$\delta ny = -\tfrac{1}{2}\frac{1}{T} IA^2 \tan\theta\ (\sin 2\psi),$$

i.e., the effect is proportional to $\frac{1}{T}$, $\tan\theta$ or nx; it also varies as A^2 and to each circular zone in the stop corresponds a double circle in the image plane of diameter equal to the distance of its centre from the ideal focal point.

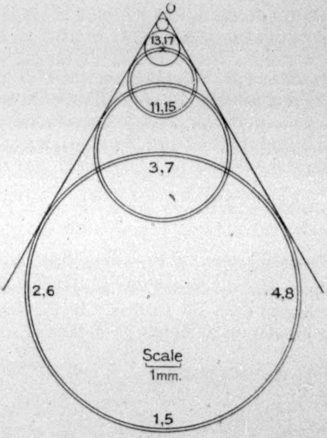

FIG. 4.

Thus, the defect leads to an unsymmetrical flare, as indicated in Fig. 4, and is called Coma of the First Order.

When spherical aberration is present the two circles no longer co-incide, and they become a double loop.

III. The terms K and G lead to

$$\delta x = -\tfrac{1}{2}T \tan^2\theta A \cos\psi . \quad (2K+G),$$
$$\delta y = -\tfrac{1}{2}T \tan^2\theta A \sin\psi . \quad G,$$

so that the circle on the stop corresponds to an ellipse on the focal plane whose axes are in the ratio of $2K+G : G$; when $K=O$, the ellipse becomes a circle, and the defect being proportional to A is of the same type as that due to an error in focus$=-\tfrac{1}{2}P\tan^2\theta G$. When K is not O the effect is that two focal lines are produced at distances from the focal plane

$$= -\tfrac{1}{2}T \tan^2\theta(2K+G),$$

and $\qquad -\tfrac{1}{2}T \tan^2\theta G.$

K thus determines the astigmatism of the lens. The mean value of $-\tfrac{1}{2}T\tan^2\theta\ (K+G)$ determines the distance of the best average focus from the focal plane. The term $G-K=\Sigma P_5/T$, being independent of separations except in so far as they affect T, is simply calculated, and is often referred to as Petzval's expression.

IV. The coefficient H, being independent of A, must represent a departure of the image from its theoretical position, and is called the distortion; its effect is proportional to the cube of $\tan\theta$, and it leads to the defect of unequal magnification for objects of different size, thus making straight lines appear to be curved.

The curvature of any straight line is $-H\tan\theta_1 T$, where θ_1 is the least angular value of θ for points on the straight line.

Formulæ referred to Vertices of Surfaces.

Exactly analogous results are obtained by considering all distances measured from the surface. If the same method of treatment be adopted we obtain the following results :—

Let

$$F_1=(n_2-n_0)R_1\,; \qquad \frac{B_1B_2}{n_2}=d_2 .$$
$$\frac{B_1A_0}{n_0}=u_1\,; \qquad \frac{B_1A_2}{n_2}=v_1 .$$
$$U_1=\frac{1}{u_1}\,; \qquad V_1=\frac{1}{v_1} .$$
$$F_1=V_1-U_1\,; \qquad \frac{l_3}{l_1}=1-d_2F_1 .$$
$$V_{13}=V_1+\frac{l_3}{l_1}F_3\,; \qquad \frac{l_5}{l_1}\ \frac{l_3}{l_1}-d_4V_{13} .$$
$$v_3=\frac{l_3}{l_1}/V_{13} . \qquad . \ . \ . \ . \ . \ . \ . \ . \ . \qquad (24)$$

The apparent distance d_0 of any point from the first surface is given by

$$\frac{d_0}{\frac{l_0}{l_1}\,\frac{l_1}{l_1}}=\sum\frac{d_4}{\frac{l_3}{l_1}\,\frac{l_5}{l_1}},$$

where the summation is taken from the first plane to the plane of the point.

For the case of the centre of curvature (5)

$$\frac{d_0}{\frac{l_0}{l_1}\,\frac{l_1}{l_1}}=\sum\frac{d_4}{\frac{l_3}{l_1}\,\frac{l_5}{l_1}}+\frac{r_5}{n_4\frac{l_5}{l_1}\,\frac{l_6}{l_1}}, \quad . \quad . \quad . \quad . \qquad (25)$$

where l_6/l_1 is the intercept at the centre of curvature.

The lateral aberration due to surface 5 is given by

$$\begin{aligned}\delta x=-\tfrac{1}{2}f\,.\,\{&e_5(n\sin A)^3\cos\psi+i_5(n\sin A)^2(1+2\cos^2\psi)\tan\theta\\&+2k_5\cos\psi(n\sin A)\tan^2\theta+g_5\cos\psi n\sin A\tan^2\theta\\&+h_5\tan^3\theta\}.\end{aligned}$$

$$\begin{aligned}\delta y=-\tfrac{1}{2}f\,.\,\{&e_5(n\sin A)^3\sin\psi+i_5(n\sin A)^2(2\sin\psi\cos\psi)\tan\theta\\&+g\sin\psi(n\sin A)\tan^2\theta\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)\end{aligned}$$

where n is the refractive index of the final medium in which δx and δy are measured; A, ψ and θ have the same values as in the equations (19) and (20); $f=1/V$, and

$$e_5=\frac{l_5}{l_1}\left(\frac{V_{135}}{n_6^2}-\frac{V_{13}}{n_4^2}\right)\left\{\frac{1}{D_5'F}\right\}^2,$$

$$i_5=D_5e_5 \qquad k_5=D_5i_5,$$

$$g_5-k_5=P_5/V=R_5\left(\frac{1}{n_4}-\frac{1}{n_6}\right)/V,$$

$$h_5=D_5g_5.$$

$$\frac{1}{D_5'V}=+\frac{n_6l_6}{r_5l_1}\,\frac{1}{V}=-\left(\frac{V_{135}}{n_6}-\frac{V_{13}}{n_4}\right)\Big/\left(\frac{1}{n_6}-\frac{1}{n_4}\right)V,$$

and $$D_5V=\frac{l_1}{l_5}D_5'V+\sum_2^4\frac{d_4V}{\frac{l_3}{l_1}\cdot\frac{l_5}{l_1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

The value of D_5V may also be expressed as

$$\frac{l_1}{l_5}D_5'V+\left(\frac{L_5}{L}\Big/\frac{l_5}{l_1}-\frac{L_1}{L}\right), \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

where L_1, L_5 L are quantities similar to l_1, l_3, l_5 calculated for the ray incident parallel to the axis and in the reversed direction.

By either of the methods given already the Aberrations of the First Order can be calculated for any definite object position. When the lens system has to be used for different magnifications it is preferable to make use of the method of the characteristic function, either calculating the coefficient directly or using the above results to deduce the coefficients of the characteristic function.

The annexed calculation shows the practical application of the formulæ given above; the logarithms are omitted when the calculation is quite straightforward, but inserted where the arrangement of the calculations is important.

[*Added February* 9.

The following method of expressing the value of p in terms of the co-ordinates of the rays is often useful :—

To the approximation considered p_5 may be replaced by the optical length of the intercept on the central plane of 5 by the ray.

Consider the images in medium 6 of the object plane (0) and the stop plane (1), and let the ray considered meet these planes in P_6, Q_6, and the central plane in K_6. Let the corresponding axial points be A_6 and B_6. Let this ray meet the object plane in $[x_0, y_0]$ and the stop plane in $[x_1, y_1]$, then the ray through x_0, y_0 on the object plane and o, o on the stop plane will pass through P_6 and B_6 to the first approximation, while the ray through o, o on the object plane and $[x_1, y_1]$ on the stop plane will pass through A_6 and Q_6 to the same approximation.

Let these rays meet the central plane in K_6' and K_6'',

then $$C_5K_6=C_5K_6'+C_5K_6''.$$

If q_1, q_3, q_5, &c., be calculated for the paraxial ray through o, o meeting the stop at q_1 from the axis, we have

$$[C_5K_6']=[x_1, y_1]\frac{q_5}{q_1}.$$

Aberrations of Photographic Lens.

Object at infinity. $S=O$.

$r_1=-0\cdot128965$	$d_0=0\cdot01583$	$n_0=1$	log.
$r_3=-0\cdot049597$	$d_2=0\cdot01277$	$n_2=1\cdot51170$	$=0\cdot179466$
$r_5=+0\cdot196423$	$d_4=0\cdot00664$	$n_4=1\cdot54780$	$=0\cdot189715$
$r_7=-0\cdot126663$	$d_6=0\cdot02114$	$n_6=1\cdot61250$	$=0\cdot207500$
		$n_8=1$	

$$P_1=\frac{n_2-n_0}{n_2n_0}\frac{1}{r_1}=-2\cdot62468 \qquad a_0=(d_0+r_1)\,.\,n_0=-0\cdot113135$$

$$P_3=\frac{n_4-n_2}{n_4n_2}\frac{1}{r_3}=-0\cdot311079 \qquad a_2=n_2(-r_1+d_2+r_3)=+\cdot0139285$$

$$P_5=\frac{n_6-n_4}{n_6n_4}\frac{1}{r_5}=+0\cdot131976 \qquad a_4=n_4(-r_3+d_4+r_5)=+0\cdot391067$$

$$P_7=\frac{n_8-n_6}{n_8n_6}\frac{1}{r_7}=+2\cdot99887 \qquad a_6=n_6(-r_5+d_6+r_7)=-0\cdot486889$$

	Log.	No.			No.
P_1q_1……	0·419077−	−2·62468	$1\cdot000=q_1$	$P_1/T_{1\cdot3\cdot5\cdot7}$	−2·62460
$-a_2$ ….	$\bar{1}$·143905−			$P_3/T_{1\cdot3\cdot5\cdot7}$	−0·31107
	$\bar{1}$·562982+		=0·36558	$P_5/T_{1\cdot3\cdot5\cdot7}$	+0·13197
	0·135317		$1\cdot36558=q_3$	$P_7/T_{1\cdot3\cdot5\cdot7}$	+2·99878
P_3………	$\bar{1}$·492871−				
P_3q_3 ….	$\bar{1}$·628188−	−0·42480		a_0	−0·113135
	0·484226−	$-3\cdot04948=T_{1\cdot3}$		$a\,T/q_1q_3$	+0·101994
$-a_4$ ……	$\bar{1}$·592251−			Σ ,,	−0·011141
	0·076477+		=1·19255	a_4T/q_3q_5	+0·111944
	0·407922+		$2\cdot55813=q_5$	Σ ,,	+0·100803
P_5 …….	1·120496+			a_6T/q_5q_7	−0·153763
P_5q_5 ….	$\bar{1}$·528418+	+0·33761		Σ ,,	−0·052960
$T_{1\cdot3\cdot5}$ …	0·433269−	$-2\cdot71187=T_{1\cdot3\cdot5}$			
$-a_6$ ….	$\bar{1}$·687429+				
	0·120698−		=−1·32035		
	0·092643+		$+1\cdot23778=q_7$		
P_7 …….	0·476957+				
P_7q_7……	0·569600+	+3·71190			
	0·000013	$+1\ 00003=T_{1\cdot3\cdot5\cdot7}$			

q_1 is taken as 1, so that the quantities $\frac{q_3}{q_1}\frac{q_5}{q_1}$, &c., may be denoted by q_3, q_5, &c.

—	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Aberration coefficients for complete lens.
P_1q_1	0·419077—	...	$\bar{1}$·628188—	...	$\bar{1}$·528418+	...	0·569600+	...	...
$(Pq)^3$	1·257231—	...	$\bar{2}$·884564—	...	$\bar{2}$·585254+	...	1·708800+	...	...
q_1	0·000000	...	0·135317+	...	0·407922+	...	0·092643+	...	...
$1/n_2-n_0$	0·290985+	...	1·442493+	...	1·189096+	...	0·212894—	...	...
n_2n_0/n_2-n_0 .	0·470451+	...	1·811674+	...	1·586311+	...	0·420394—	...	...
	2·018667—	—104·392	2·274048—	—187·953	1·768583+	+58·6926	2·434731+	+272·102	...
P_1q_1	0·419077—	...	$\bar{1}$·628188—	...	$\bar{1}$·528418+	...	0·569600+	...	...
S_1	...	0	0·419077—	...	0·484226—	...	0·433269—	...	...
T_1	0·419077	...	0·484226—	...	0·433269—	...	0·000013+	...	...
q_1	...	...	0·135317+	...	0·407922+	...	0·092643+	...	...
	...	0	0·666808—	—4·64310	0·853835+	+7·1422	1·095525—	—12·460	...
E	2·018667—	—104·392	2·263186—	—183·310	1·712232+	+51·5504	2·454177+	+284·562	+48·411
$\Sigma a/q\,.\,q$	$\bar{1}$·053597—	...	$\bar{2}$·046924—	...	$\bar{1}$·003473+	...	$\bar{2}$·723948—	...	...
I	1·072264+	+11·8104	0·310110+	+2·04226	0·715705+	+5·1964	1·178125—	—15·0704	+3·9787
	$\bar{1}$·053597—	...	$\bar{2}$·046924—	...	$\bar{1}$·003473+	...	$\bar{2}$·723948—	...	...
K	0·125861—	—1·33617	$\bar{2}$·357034—	—0·02275	$\bar{1}$·719778+	+0·52381	$\bar{1}$·902073+	+0·79813	—0·03698
P_1/T	...	—2·62460	...	—0·31107	...	+0·13197	...	+2·99878	...
G	0·597780—	—3·96077	$\bar{1}$·523512—	—0·33382	$\bar{1}$·816758+	+0·65578	0·579430+	+3·79691	—0·15810
	$\bar{1}$·053597—	...	$\bar{2}$·046924—	...	$\bar{1}$·003473+	...	$\bar{2}$·723948—	...	...
H	$\bar{1}$·651377+	+0·44810	$\bar{3}$·570436+	+0·003719	$\bar{2}$·820231+	+0·066105	$\bar{1}$·303378—	—0·20108	+0·31684

If q_0', q_3', q_5', &c., be calculated for the paraxial ray through o, o, meeting the object plane at q_0' and the central planes at q_3', q_5', &c., we have

$$[C_5K_6'']=[x_0, y_0]\frac{q_5'}{q_0'},$$

and

$$[C_5K_6]=[x_1, y_1]\frac{q_5}{q_1}+[x_0, y_0]\frac{q_5'}{q_0'}.$$

And this quantity can be substituted for p_5 in equation 16.

ABSTRACT.

The Paper describes approximate methods of treatment of the first order aberrations of a centred optical system. Two methods are used, one primarily suited to the case where the separation of the surface is small, and the other more suited for use where the separations vitally affect the design.

The aberrational defects are expressed as lateral aberrations—*i.e.*, as defects measured in the focal plane of the system. The procedure adopted is to express the aberrational defect of a single surface in terms of the constants of the surface, and the perpendicular distance of the ray considered from the centre of curvature of the surface. The value of this perpendicular can be expressed in terms of the co-ordinates of the ray in any chosen medium, and thus the aberration due to each surface can be expressed in terms of the co-ordinates of the chosen ray, in such a way that the aberrations of the individual surfaces can be summed.

DISCUSSION.

Mr. T. **Smith** congratulated the author on his Paper, which he considered an interesting and valuable addition to the existing discussions on optical aberrations. He had himself used a similar reference system in dealing with this subject, but in place of a partially geometrical discussion had preferred to use a purely analytical method. The geometrical treatment was obviously advantageous in giving an insight into the mysteries of aberration to students, but it was very difficult to be certain that every contributory cause had been considered. The results stated in the Paper were correct, but he was not clear that one possible contributory cause, which happened to this approximation to give a zero contribution, had received adequate consideration. If in a system of refracting surfaces at some intermediate theoretical image plane an aberration δy were present, the corresponding effect in the final image plane should not be assumed to be $m\delta y$, where m is the first order magnification which the image subsequently undergoes. The exact co-ordinates to the second approximation $y+\delta y$, and the exact magnification to the same order $m+\delta m$ should be considered, so that the final co-ordinates to this order are $my+y\delta m+m\delta y$. It can be shown that when all the surfaces are taken into consideration the contribution of the type $y\delta m$ is zero. He thought that the Paper could with advantage consider this point more fully. An interesting question which arose in connection with the analysis of refraction into displacements according to the ordinary laws of geometrical optics with aberrations superposed, related to the limits within which such a procedure was justifiable. The ordinary

laws could only be taken as a possible first approximation for all rays for refractive indices exceeding 3. Instead of an incident angle in air of a right angle, the first approximation may only be used up to an incident angle of cos $^{-1}\frac{3-\mu}{2\sqrt{2}}$. As the index falls in value the range over which the approximation holds becomes less, and in the limiting case, when $\mu=1$, the range is from 0° to 45° compared with the range 0° to 90° over which refraction takes place.

Prof. LEES said that these methods appeared to start off from the approximate assumption that the image of a plane was a plane. Did not this lead to errors comparable with those which were being dealt with ?

Mr. CHALMERS replied as follows : The criticisms made by Mr. Smith are not, I think, justified, because the terms to which he refers do not arise in the method of treatment adopted. In transferring the aberration to the last medium it is permissible to multiply by the paraxial magnification since the actual point of crossing the object plane is transferred to the image plane by multiplying its co-ordinates by the paraxial magnification, and adding aberrational defects for this surface for the actual ray. This aberrational defect is expressed in terms of the modified values of the co-ordinates, but as the modifying terms are multiplied by the small aberrational terms this introduces terms of higher order only. If it be necessary to consider terms of higher order, it would be necessary to take them into account in expressing values of the p's in terms of the co-ordinates in the initial medium.

Mr. Smith's remarks on limits to which approximation could be applied are very interesting : they show that approximations are justified in practically all cases which arise, except for a few cases of refraction at plane surfaces, and in this case the term he quotes is multiplied by a zero curvature.

IX. *Note on the Use of Approximate Methods in Obtaining Constructional Data for Telescope Objectives.* By T. Smith, *B.A.* (*From the National Physical Laboratory.*)

Received January 19, 1917.

In earlier communications* the author has described a method of calculating cemented telescope objectives when thicknesses are entirely neglected. The effects of the thicknesses of actual objectives on the aberrations are distinctly appreciable in spite of the small values of the thicknesses in comparison with the focal length of the objective. This, however, so far from invalidating the calculations, is an essential feature of the method employed. The reason is that the second order aberrations are always negative, while the introduction of thicknesses causes the first order aberrations to rise from zero to some positive value. To obtain a satisfactory objective a balance must be struck between a positive first order aberration and a negative second order aberration, and the greater the relative aperture at which the objective is to be used the greater must be the amount of first order aberration introduced to effect the most satisfactory compromise. The larger the aperture, however, the larger must the thicknesses of the lenses be, so that without altering the curvatures of the surfaces increasing amounts of first order aberration will be automatically secured. It is a fortunate coincidence that the amounts of aberration introduced by augmenting the thicknesses are of the order required to give a satisfactory objective of the aperture these thicknesses will yield. This result, of course, only holds within limits; but these limits happen to cover the region within which small objectives are normally constructed. Many computers do not appear to be aware of this automatic compensation, but have abandoned the use of approximate methods in calculating simple objectives in favour of trigonometrical ray tracing. They have probably in their approximate methods taken account of the thicknesses to be given to the lenses, and solved the equations for the removal of first order aberrations with these additional terms present. Such a method does undoubtedly lead to objectives which require appreciable correction. The simpler method, on the other hand, yields objectives which require no alteration or trigonometrical verification, provided the conditions are not quite abnormal. It must be understood that this observation applies only to the particular class of objective now considered, though extensions of the application of approximate methods as satisfactory final methods of calculation are to be expected as a consequence of the systematic exploration of higher order aberrations in other types of optical instrument.

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* Proc. Phys. Soc. XXVII., p. 485; XXVIII., p. 220; XXX., p. 31.

The automatic compensation for thicknesses in the case of telescope objectives is of sufficient interest and importance to merit theoretical investigation, and that is the object of this note. A simple demonstration will show that the compensation may hold for a considerable range of apertures. Consider any thin objective which has such curvatures that the first order spherical aberration is zero. Without altering the curvatures of the surfaces, let small but finite thicknesses be given to the various component lenses. A series of objectives of different thicknesses can be imagined in which the ratios of the thicknesses of the various component lenses to one another are constants throughout the series. A single variable, t, proportional to the thicknesses of a typical member of the series, will then suffice to determine that member. When t is zero the first order spherical aberration coefficient is zero. When t is not zero but small, the first order spherical aberration coefficient will in general be proportional to t; let it be denoted by αt, where α is a constant for all lenses of the series considered. As has been pointed out above, α is essentially positive. The second order spherical aberration coefficient is for these objectives negative, and, when the lens is thin, may be denoted by $-\beta$. When thicknesses are introduced there will be a correction to be added to β containing t as a factor; but, as β is a large positive number, the thickness correction to it is negligible, and the second order coefficient may thus be taken to be equal to $-\beta$ for all lenses of the series. If the third and higher order aberrations may be neglected, the total longitudinal aberration of the typical objective is

$$\alpha t P^2 - \beta P^4,$$

where P is the relative semi-aperture for the ray considered. Now, if Q is the maximum semi-aperture attainable in this objective, t is proportional to Q^2. The aperture P for which correction is desired is a constant fraction of the full aperture Q. Thus correction should be effected for the aperture P where $P^2 = \gamma t$, γ being positive and constant for all members of the series. On substituting this value for P^2, the longitudinal aberration is seen to be

$$(\alpha-\beta\gamma)\gamma t^2,$$

and this will vanish for all members of the series if

$$\alpha=\beta\gamma.$$

This result shows that the compensation may be automatically effected over an appreciable range of apertures. The extent of the range is dependent, among other factors, on the third and higher order aberrations.

If the thin objective is corrected for first order coma, a similar argument shows that there may be automatic compensation for this aberration also. It will not, however, ordinarily be the case that the spherical aberration and coma are corrected in this way for the same aperture. As a rule the effect of thicknesses on coma can be safely neglected. The conclusions to which this argument leads may be put into the general form :

" If a thin lens is free from first order spherical aberration or coma, the introduction of thicknesses without any alteration in the curvatures of the surfaces yields objectives which are corrected for a zone bearing a constant ratio to the maximum aperture obtainable with the thicknesses introduced, provided effects proportional to the sixth and higher powers of the maximum aperture may be neglected."

The problem may now be investigated in some detail. The case of a cemented doublet may be taken as an illustration. It would be easy to work out the effects of the thicknesses on such a system by taking the curvatures of the thin objective, inserting thicknesses and calculating the aberrations of the new system from the fundamental formulæ. This course involves the introduction of more variables than are necessary for the discussion of the problem, and an alternative method is therefore followed.

The spherical aberration σ and coma σ' of any system are expressed in terms of the six aberration coefficients, A, B, C, ϖ, B', A', by the formulæ

$$\sigma=A-m(4B+1)+2m^2(3C+\varpi)-m^3(4B'+1)+m^4A',$$

and

$$\sigma'=A-m(3B+1)+m^2(3C+\varpi)-m^3B'$$
$$-s\{B-m(3C+\varpi)+m^2(3B'+1)-m^3A'\},$$

where m is the magnification for the object and s the magnification for the aperture stop. When the objective is "thin" the relations

$$A-2B-1+C=B-2C-\varpi+B'=C-2B'-1+A'=0$$

are satisfied, so that

$$4\sigma=(1-m)^4\{4C+2\varpi+1-4(B'-B)M+(2\varpi+3)M^2\}$$

and

$$4\sigma'=(1-m)^3(1-s)\{4C+2\varpi+1-(B'-B)(S+3M)$$
$$+\varpi M(S+M)+M(2S+M)\},$$

where

$$M=\frac{1+m}{1-m},\ S=\frac{1+s}{1-s}.$$

For the thin objective the conditions for freedom from spherical aberration and coma thus involve

$$4C+2\varpi+1=(5+2\varpi)M^2$$

and

$$B'-B=(2+\varpi)M.$$

The actual thick objective may be regarded as a combination of three systems; the first being a meniscus single lens of the same material as the front component of the thin doublet, with a specified thickness, and having the curvature of each surface equal to that of the first surface of the doublet; the second the thin doublet; the third a single meniscus lens of the same glass as the back component of the doublet, with a given thickness, and each surface equal in curvature to that of the last surface of the doublet. If suffixes are used to distinguish the components and the magnifications at which they are operating, the laws for the addition of the aberrations of a series of systems may be expressed in the form

$$f_{1,n}\sigma_{1,n}=\sum_1^n f_\lambda\sigma_\lambda m^4{}_{\lambda+1,n}$$

$$f_{1,n}\sigma'_{1,n}=\sum_1^n f_\lambda\sigma'_\lambda m^3{}_{\lambda+1,n}\,s_{\lambda+1,n},$$

where f is the focal length of the lens or element determined by its suffix.

Let κ denote the power of the first surface of the thin doublet, the refractive index being μ. The first component is the meniscus lens of the same glass of thickness t and surfaces of powers κ and $-\kappa$ respectively. The general formulæ for the aberration coefficients at once give for the first component

$$A(\mu-1)^2\kappa^2t^2=(\mu+1)^2-2\kappa t(\mu^2+\mu+1)+\kappa^2t^2(\mu+1)^2-\kappa^3t^3\mu,$$
$$B(\mu-1)^2\kappa^2t^2=(\mu+1)^2-\kappa t(\mu^2+\mu+1)+\kappa^2t^2\mu,$$
$$C(\mu-1)^2\kappa^2t^2=(\mu+1)^2,$$
$$B'(\mu-1)^2\kappa^2t^2=(\mu+1)^2+\kappa t(\mu^2+\mu+1)+\kappa^2t^2\mu,$$
$$A'(\mu-1)^2\kappa^2t^2=(\mu+1)^2+2\kappa t(\mu^2+\mu+1)+\kappa^2t^2(\mu+1)^2+\kappa^3t^3\mu,$$
$$\varpi=0.$$

In the last component the power of the first surface has the opposite sign to that of the last surface of the doublet. Similar formulæ will thus apply to this component if accents are added and the sign of κ is changed, if the back lens of the thin doublet is of refractive index μ' with the last surface of power κ', and t' is the thickness to be introduced.

The powers of the three components of which the thick lens is supposed to be composed are

$$\kappa_1=\kappa^2t/\mu, \qquad \kappa_2=1, \qquad \kappa_3=\kappa'^2t'/\mu',$$

with

$$\kappa_{1,2}=1-\kappa(1-\kappa)t/\mu, \qquad \kappa_{2,3}=1-\kappa'(1-\kappa')t'/\mu',$$

and

$$\kappa_{1,3}=1-\kappa(1-\kappa)t/\mu-\kappa'(1-\kappa')t'/\mu'+\kappa\kappa'(1-\kappa-\kappa')tt'/\mu\mu'.$$

The partial magnifications are given in terms of m, the magnification for the complete system by

$$m_1=\frac{\kappa_{1,3}m}{\kappa_1+\kappa_{2,3}m}, \qquad m_2=\frac{\kappa_1+\kappa_{2,3}m}{\kappa_{1,2}+\kappa_3m}, \qquad m_3=\frac{\kappa_{1,2}+\kappa_3m}{\kappa_{1,3}}.$$

If only the first powers of t and t' are retained these may be written

$$m_1=1-\frac{\kappa t}{\mu}p,$$
$$m_2=m\left(1+\frac{\kappa t}{\mu}p-\frac{\kappa' t'}{\mu'}q\right),$$

and

$$m_3=1+\frac{\kappa' t'}{\mu'}q,$$

where

$$p=1-\kappa+\kappa/m,$$

and

$$q=1-\kappa'+\kappa' m.$$

Now from the previous formulæ

$$\sigma_1=\frac{1}{(\mu-1)^2\kappa^2t^2}\left[(\mu+1)^2(1-m_1)^4-2\kappa t(\mu^2+\mu+1)(1-m_1)^2\right.$$
$$\left.(1-m_1{}^2)+\kappa^2t^2(\mu+1)^2(1-m_1)(1-m_1{}^3)-\kappa^3t^3\mu(1-m_1{}^4\right]$$
$$=\frac{\kappa^2t^2p(p-\mu)}{(\mu-1)^2\mu^4}\left[(\mu+1)^2p^2-\mu(3\mu^2+2\mu+3)p+4\mu^3\right],$$

and similarly

$$\sigma_3=\frac{\kappa'^2t'^2q(q-\mu')}{(\mu'-1)^2\mu'^4}\left[(\mu'+1)^2q^2-\mu'(3\mu'^2+2\mu'+3)q+4\mu'^3\right],$$

when the lowest powers of t and t' only are retained. The direct contribution of the first component to $/_{1,3}\sigma_{1,3}$ is, therefore, to the order required

$$\frac{tm^4p(p-\mu)}{(\mu-1)^2\mu^3}\left[(\mu+1)^2p^2-\mu(3\mu^2+2\mu+3)p+4\mu^3\right],$$

and that of the last component is

$$\frac{t'q(q-\mu')}{(\mu'-1)^2\mu'^3}\left[(\mu'+1)^2q^2-\mu'(3\mu'^2+2\mu'+3)q+4\mu'^3\right].$$

The second component has been calculated to be free from aberration for the magnification m, but it is now working at the different magnification m_2. If sigmas without suffixes relate to the thin lens as calculated, the condition $\sigma=0$ requires the coefficients of the doublet to have the values

$$B'-B=(2+\varpi)M-\frac{2\sigma'}{(s-m)(1-m)^2},$$

and

$$4C+2\varpi+1=(5+2\varpi)M^2-\frac{8\sigma'M}{(s-m)(1-m)^2},$$

σ' being the coma coefficient of the thin objective.

For the magnification m_2 the spherical aberration of the thin objective is, therefore, given by

$$\sigma_2=m\left(\frac{\kappa t}{\mu}p-\frac{\kappa't'}{\mu'}q\right)\left(\frac{4\sigma'}{s-m}+m^2-1\right)$$

since

$$M_2=\frac{1+m_2}{1-m_2}=M+\frac{2m}{(1-m)^2}\left(\frac{\kappa t}{\mu}p-\frac{\kappa't'}{\mu'}q\right).$$

The value of the spherical aberration coefficient for the lens with thicknesses introduced is thus, to the first powers of the thicknesses,

$$\sigma_{1,3}=\frac{tm^4p(p-\mu)}{(\mu-1)^2\mu^3}\,[(\mu+1)^2p^2-\mu(3\mu^2+2\mu+3)p+4\mu^3]$$
$$+m\left(\frac{\kappa t}{\mu}p-\frac{\kappa' t'}{\mu'}q\right)\left(\frac{4\sigma'}{s-m}+m^2-1\right)$$
$$+\frac{t'q(q-\mu')}{(\mu'-1)^2\mu'^3}\,[(\mu'+1)^2q^2-\mu'(3\mu'^2+2\mu'+3)q+4\mu'^3].$$

The most important case is $m=0$. The value of mp is then equal to κ, and

$$\sigma_{1,3}=\frac{t\kappa^4(\mu+1)^2}{(\mu-1)^2\mu^3}+\frac{t\kappa^2}{\mu}\left(\frac{4\sigma'}{s}-1\right)$$
$$+\frac{t'q(q-\mu')}{(\mu'-1)^2\mu'^3}\,[(\mu'+1)^2q^2-\mu'(3\mu'^2+2\mu'+3)q+4\mu'^3].$$

Take as a numerical example the ordinary form of objective composed of hard crown and dense flint with the former leading.* The last surface is flat, so that q is unity and the contribution of the third element to $\sigma_{1,3}$ is thus

$$\frac{-t'(\mu'^2-1)}{\mu'^3},$$

the ordinary formula for a thick plate. With the glass proposed its value is very approximately $-\cdot38t'$. The value of κ is about 1·25,* and the contribution of the first element is, therefore, approximately equal to $16{\cdot}5t$. When $s=1$, $\sigma'=-1{\cdot}47$,* so the contribution of the doublet is $-7{\cdot}1t$. As a rule t' is about half as large as t, so the total value is approximately $9{\cdot}2t$. The second order spherical aberration coefficient for the thin lens is about -107, so to the second order the spherical aberration with crown thickness t is

$$\tfrac{1}{2}\times 9{\cdot}2tP^2-\tfrac{1}{8}\times 107P^4,$$

where P is the relative semi-aperture. The zone for which the aberration is corrected is thus given by $P^2=0{\cdot}34t$. The maximum semi-aperture the crown lens can have for this thickness is Q where $Q^2=0{\cdot}42t$. Thus $P=0{\cdot}9Q$ approximately, a value quite satisfactory for most purposes. The neglected aberrations tend to reduce this value somewhat when the aperture is large, the correction thus remaining excellent over a long range.

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* See "Constructional Data of Small Telescope Objectives." Smith & Cheshire. Pp. 8 and 9.

As the refractive index of the crown glass approaches that of the flint, κ becomes smaller and σ', though still negative, becomes smaller in absolute amount. The total contribution of the elements remains positive, but falls in value. The second order aberration coefficient also becomes less in absolute amount, and the tendency for the correction to be automatically secured for a reasonable zone compared with the full aperture attainable remains.

ABSTRACT.

The Paper discusses the reason why satisfactory telescope objectives are obtained by neglecting thicknesses and solving for freedom from first order aberrations. It is shown that the introduction of thicknesses into such an objective without any alteration in the curvatures of the surfaces yields a lens corrected for aberration for a zone which is a constant fraction of the full aperture obtainable. For objectives of the usual type this zone is very approximately the one that would be selected for correction to obtain the most favourable balance between first and second order aberrations. It follows that objectives calculated from first order formulæ in which thicknesses are neglected do not require trigonometrical verification or correction unless the conditions are very abnormal.

DISCUSSION.

Prof. J. W. NICHOLSON said that the author had brought out an important point in this Paper. There was an idea prevalent in many quarters that the thin lens was simply an abstraction of no practical importance, but apparently this was by no means the case. He had been familiar in a general way with this compensation of first and second order aberrations, but had never seen it mentioned in any publication. Were there any convenient circumstances in which both coma and spherical aberration are simultaneously compensated for the same aperture ?

Prof. C. H. LEES asked how the magnitude of the residual aberration or the shape of the aberration curve altered when the zone of correction was altered by changing the index of the crown glass.

The AUTHOR, in reply to Prof. Nicholson, said he had not worked out numerical cases for coma, though the formulæ were quite simple. In practice, it is usually found that in the absence of spherical aberration if the coma is approximately corrected the result is satisfactory. He had no doubt, however, that with triple objectives both coma and spherical aberration could be compensated for the same aperture. In reply to the President, generally speaking, with very low indices of the crown the second order aberration is large, while with indices nearer the flint it is smaller. The ratio of extreme cases might be about 5 to 1.

X.—*A Suggestion as to the Origin of Spectral Series.* By

H. STANLEY ALLEN, *M.A., D.Sc.*

RECEIVED JANUARY 25, 1918.

CONSIDER an electron (charge e, mass m) moving in a circular orbit of radius r about the centre of the core of an atom. Neglecting for the present any effect due to a magnetic field, the equation of motion is

$$mr\omega^2 = Ee/r^2,$$

or

$$mr^3\omega^2 = Ee, \quad \ldots \ldots \quad (1)$$

where ω is the angular velocity, and E the charge attracting the electron.

We now assume that the principle of the constancy of angular momentum may be applied to the *total* angular momentum of the electron and a certain part of the core bearing a special relation to the electron. This angular momentum is measured with regard to the axis of the core, which provides a line of reference about which the motion takes place. The difference between this assumption and that made by Bohr must be noticed. Bohr assumes that the electron and the nucleus rotate round the common centre of gravity, and the total angular momentum of electron and nucleus is then $\tau h/2\pi$ where τ is an integer and h is Planck's constant. The new assumption gives

$$mr^2\omega \mp I\,\Omega = \tau h/2\pi,$$

or

$$mr^2\omega = (\tau \pm \mu)h/2\pi, \quad \ldots \ldots \quad (2)$$

where

$$\mu = 2\pi I\,\Omega/h \quad \ldots \ldots \quad (3)$$

Here I denotes the moment of inertia, Ω the angular velocity of part of the core.

Thus μ may be regarded as proportional to the angular momentum of a definite portion of the core.* The assumption made is admittedly an arbitrary one, but so is any assumption as to the constancy of angular momentum. This constancy may be regarded as a condition imposed upon the motion of the electron by the " binding " of the tube of force connecting it to the core, but there is something more to be said for the present point of view inasmuch as the connection between electron and core is now partly dependent on the magnetic field, which is regarded as imparting stability to the electronic system. Thus it is appropriate to take the magnetic axis of the core as the axis for calculating the angular momentum.

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* H. S. Allen, Phil. Mag., Vol. XXIX., p. 714, 1915.

From equations (1) and (2) we find as the values of r and ω,

$$r=\frac{(\tau\pm\mu)^2h^2}{4\pi^2mEe}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\omega=\frac{8\pi^3(Ee)^2m}{(\tau\pm\mu)^3h^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The energy, W, necessary to remove the electron to an infinite distance is given by

$$W=\frac{Ee}{2r}=\frac{2\pi^2m(Ee)^2}{(\tau\pm\mu)^2h^2}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In the passage from one "stationary" state to another monochromatic radiation is supposed to be emitted, and the frequency, ν, is determined by the relation

$$\begin{aligned} h\nu=\delta W&=W_2-W_1 \\ &=\frac{2\pi^2m(Ee)^2}{h^2}\left(\frac{1}{(\tau_2\pm\mu_2)^2}-\frac{1}{(\tau_1\pm\mu_1)^2}\right). \quad (7)\end{aligned}$$

Putting $E=ke$, and $\frac{2\pi^2me^4}{h^3}=\nu_0$, the equation may be written

$$\nu=k^2\nu_0\left(\frac{1}{(\tau_2\pm\mu_2)^2}-\frac{1}{(\tau_1\pm\mu_1)^2}\right) \quad . \quad . \quad . \quad . \quad (8)$$

Here ν_0 is the frequency corresponding to Rydberg's constant. Using wave numbers instead of frequencies, we obtain

$$N=k^2N_0\left(\frac{1}{(\tau_2\pm\mu_2)^2}-\frac{1}{(\tau_1\pm\mu_1)^2}\right). \quad . \quad . \quad . \quad (9)$$

Rydberg's constant, N_0, appears here as a universal constant having exactly the same value for all elements. This is in agreement with the conclusion of Nicholson* from a critical examination of the arc spectrum of helium, where the constant has the same value as was obtained by Curtis for hydrogen. It has been suggested by Bohr† that the series of lines observed by Fowler and Evans, and attributed to helium, require a slightly different value for the constant, but this cannot be regarded as definitely established.

In ordinary spectra the numerical factor k must be taken as unity. When $k=2$, the Rydberg constant is multiplied by 4, and the series give the enhanced lines in spark spectra, as shown by Fowler.*

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* Nicholson, Proc. Roy. Soc., A, Vol. XCI., p. 255, 1915.

† Bohr, "Nature," Vol. XCII., p. 231, 1913.

Equation (9) represents Rydberg's formula for spectral series, the quantity μ being termed the " phase " of the series. It is important to notice that in the observed series μ_1 and μ_2 are not equal to one another. If the present interpretation be correct this implies that the angular momentum of the part of the core associated with the electron is different in the two types of state. This may arise from an actual change in the angular momentum of a particular part of the core, or it may indicate that the electron becomes attached to a different part of the core. *Whatever interpretation be given to the phase, the two types of state concerned must be in some way different from one another.*

In an earlier paper† I have shown that when the core is regarded as equivalent to a magnet of moment M a formula of the type given by Ritz can be obtained for the frequency. This may be written

$$N=k^2N_0\left\{\frac{1}{\left[\sigma_2+\frac{B}{\sigma_2^2}\right]^2}-\frac{1}{\left[\sigma_1+\frac{B}{\sigma_1^2}\right]^2}\right\}, \quad . \quad . \quad (10)$$

where

$$B=\frac{16\pi^3 mMEe^2}{h^3}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Here σ was regarded as an exact integer, but if we now adopt the suggestion that $\sigma=\tau+\mu$, where τ is integral, the formula becomes identical with Ritz's modification of Rydberg's formula. It must be pointed out, however, that the values of B deduced from observation are, in general, too large to be explained as due entirely to the action of the magnetic field of the core.

In the foregoing pages it has been assumed that in the case of a terrestrial atom, as distinguished from the simpler atoms of the nebulæ, the core is large enough to be able to produce an appreciable magnetic field. To say that the core cannot produce such a magnetic field on account of the small size of the nucleus is to beg the question at issue, for the smallness of the nucleus was first deduced by considering only electrostatic forces, whilst ignoring the possible effect of magnetic forces, in the scattering of α particles. The second reason for postulating a small nucleus is in order to account for the mass of the atom as purely electromagnetic in origin. But this assumes that the nucleus is *simple* and not *complex*. The phenomena of radio-activity point to the presence of α and β particles in the core as separate entities. It is only the ultimate positive entities (perhaps the hypothetical positive electrons of Prof. Nicholson) contained in the core, which need be considered very small in order to give the necessary inertia.

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* Fowler, Phil. Trans. Roy. Soc., A, Vol. CCXIV., p. 225, 1914.

† Phil. Mag., Vol. XXIX., p. 40, 1914.

ABSTRACT.

The note gives a development of an idea put forward in an earlier Paper, describing an atomic model with a magnetic core. It is assumed that the principle of the constancy of angular momentum may be applied to the *total* angular momentum of the electron, and a certain part of the core bearing a special relation to the electron. On the lines of Bohr's theory this leads to an expression for the oscillation frequency, which is similar to Rydberg's formula, and contains a constant which is the same for all elements. The "phase" μ of a "sequence" is regarded as proportional to the angular momentum of a definite portion of the core. In observed series the phases of the two sequences are not equal to one another; consequently, whatever interpretation be given to the phase, the two types of state concerned must be in some way different from one another. When the magnetic field of the core is taken into account, a formula is obtained which is identical with that of Ritz. An explanation of the series of enhanced lines in spark spectra is also suggested.

DISCUSSION.

Prof. J. W. Nicholson thought the fundamental assumption in the Paper seemed a natural one. Bohr's theory, in the case of hydrogen, assumed the angular momentum to be proportional to the integers in Balmer's series. In the case of more complex atoms we were compelled to assume that $\tau+\mu$ is the quantity to which the angular momentum is proportional. To consider this extra angular momentum to be attached to the nucleus is, of course, another assumption. It is equivalent to attaching a physical significance to tubes of force, which Sir J. J. Thomson has always tried to do. An interesting feature of the theory was that it gave the series of enhanced lines of spark spectra, and the physical interpretation of $k=2$, viz., that these are due to atoms with another electron torn off, seemed very feasible. As was pointed out in the Paper, any theory involving stationary states must allow for different types of stationary state—*i.e.*, different values of μ_1 and μ_2. The considerations by which Rutherford deduced the smallness of the nucleus were founded on the mathematical work of Darwin, and, on the assumption that the nucleus is wholly electrical are faultless. The fact that the nucleus might be magnetic was pointed out by Hicks, who showed that the scattering of α particles by atoms might be accounted for by magnetons rather than electrons.

Prof. G. W. O. Howe asked what was the reason for applying the word "phase" to μ. To an electrician phase had a very different meaning. What was the significance of the initial equations with which the Paper started out?

Prof. LEES asked if the atom suggested by Dr. Allen would give the same scattering effects as had been experimentally found ?

The AUTHOR, in reply, said the term "phase" was due to Rydberg, and was introduced before the Quantum-Theory was evolved. He did not know why the term had been selected. He was afraid he could not answer Prof. Howe's second question. These equations were fundamental to the Quantum-Theory, and to understand them fully one would have to know all about Quanta and the constitution of matter. With regard to the scattering of α particles, he thought it was possible to explain this as done by Hicks, if we take into account the magnetic action of the core. Of course, the electric action is also there and has to be taken into account as well. The results of experiments on scattering do not preclude the introduction of magnetic forces.

Communicated.—It appears that although Rydberg introduced the use of the symbol μ for the constant in his formula characteristic of a particular series, the term "phase" was first applied to the constant by Thiele (Astrophys. Journ. Vol. VI., p. 65, 1897). It is possible that the term was suggested by certain views as to the kinematical origin of spectra.

XI. *The Asymmetrical Distribution of Corpuscular Radiation Produced by X-rays.* *By* E. A. Owen, *B.A.* (*Cantab.*), *M.Sc.* (*Wales*), *University College, London.*

Received February 7, 1918.

According to the aether pulse theory of X-radiation, the distribution of the scattered radiation round a radiator is given by the relation $I_\theta = I_{\frac{\pi}{2}}(1+\cos^2\theta)$ where I_θ is the intensity of the scattered radiation in a direction making an angle θ with the primary beam. This represents a symmetrical distribution about the plane of the radiator. Experiment shows, however, that the distribution of the scattered radiation follows this relation only on the incident side of the radiator, there being a preponderance of radiation found on the emergent side. H. A. Wilson [*] has shown that it is possible to account for this asymmetry on the electromagnetic pulse theory in the case of metals, by assuming that the metals are made up of minute crystals embedded in amorphous material. The scattered radiation from a metal sheet is due partly to internal reflection from these small crystals which are orientated at random, and partly to regular scattering by the amorphous portion of the metal. He shows that in this case the scattered radiation should be distributed round the radiator according to the following relation :—

$$I_\theta = A\left(\frac{1+\cos^2\theta}{\cos\theta/2}\right) + B(1+\cos^2\theta),$$

where A and B are constants. The first term of the expression on the right hand side of the equation arises from the regular reflection of the rays at the crystal faces, and the second term from pure scattering by the amorphous material. Wilson points out that the experimental results of Crowther [†] in the case of aluminium agree well with this theory if it is assumed that half the scattering is due to the crystals in the metal, and the other half to the rest of the metal supposed to be in the amorphous state, *i.e.*, when $A = B$ in the above relation.

It is well known that the same asymmetry occurs also in the case of corpuscular radiation excited when X-rays fall on a thin plate of any material. Cooksey [‡] has shown that the excess emergent corpuscular radiation for silver and gold plates is of the order of 20 per cent., and this is approximately constant when the exciting X-radiation is varied over a wide range of penetration. Philpot * later, however, found that in general an increase in hardness of the exciting radiation corresponded with an increase of asymmetry, but the increase was very slight compared with the increase in the penetrating power of the exciting radiation.

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[*] Wilson, Phil Mag., Vol. 27, p. 383, 1914.

[†] Crowther, Proc. Roy. Soc. A., Vol. 85, p. 40, 1911.

[‡] Cooksey, Phil. Mag., Vol. 24, p. 37, 1912.

In view of the above explanation given by Wilson for the asymmetry observed in the case of X-rays, it is of interest to find if any difference can be detected between the ratio of emergent to incident corpuscular radiation when the screen from which the corpuscles are ejected, is changed from the crystalline to the amorphous state. A near approach to these two states can be obtained by using a screen of a salt in the dry and in the wet state respectively. The salt is held in the pores of a sheet of filter paper; in the dry state it is composed of innumerable small crystals, but when moistened it loses its crystalline nature and becomes amorphous, in which case the atoms are no longer grouped together in aggregates forming definite geometric configurations.

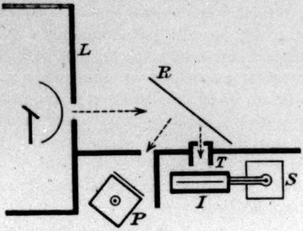

FIG. 1.

Apparatus.—The apparatus used in the investigation is sketched in Fig. 1. The rays from a platinum anticathode pass through a hole in the lead box in which the tube is enclosed, and fall on the radiator *R*. The characteristic radiation emitted by the radiator is divided into two portions, one portion passes into the primary electroscope *P*, and the other portion passes through the lead tube *T*, which cuts it down to a comparatively narrow beam. This beam enters the ionization chamber *I*, whose electrode is connected to the secondary electroscope *S*, the connecting wires being guarded by earthed tubes. The lead box and screens, and the cases of the electroscopes and the ionization chamber are also all connected to earth.

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* Philpot, Proc. Phys. Soc., Vol. 26, p. 131, 1914.

Fig. 2 shows in detail the construction of the ionisation chamber. The rays enter it through the face *P* which is a brass grid covered over with a sheet of waxed paper. The inner walls of the chamber are all covered with filter paper, so that X-rays falling on them will not give rise to an appreciable amount of corpuscular radiation. The central electrode *G* is a brass wire covered with a layer of soot, this precaution again being taken to avoid the production of corpuscular radiation, which would occur if the X-rays fell on the bare metal. The shape of the electrode is that of the chamber, rectangular in this case, and the electrode reaches within about a quarter of a centimetre of the inner walls of the

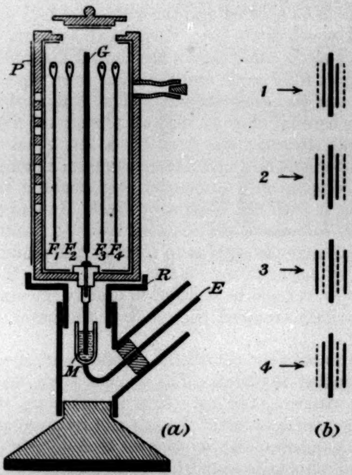

FIG. 2.

chamber. To obtain a uniform field inside the chamber, a few cotton threads rubbed with graphite were tied across the electrode.

In view of the fact that the air inside the chamber was to become laden with water vapour in the course of the experiment when the wet screens were being used, special attention was paid to the insulation of the central electrode; a plug of ambroid was used for this purpose and proved very satisfactory.

The chamber was provided with a lid by removing which the screens could be lowered into the chamber; four screens were employed which were hung from fixed supports inside the chamber, two on either side of the central electrode, of which one was pure filter paper and the other filter paper moistened with the salt to be investigated. The distance between the two inner screens, that is, the two screens on either side of the central electrode, was kept fixed and equal to 2·0 cm. The chamber was held in a fixed position in a stand, with the electrode dipping into a mercury cup attached to the wire connecting it with the electroscope leaf. It was convenient to be able to remove the chamber from its position on the stand to arrange the screens in their proper positions; after removal, by means of the bracket *R* into which the chamber fitted closely, it could be replaced in exactly the same position as it previously occupied relative to the beam of rays which entered it.

Two salts, namely, potassium bromide and silver nitrate, were examined in detail. The screens were made in the following manner. The pure filter paper screen was dipped into a saturated solution of the salt and removed after it had taken up as much of the solution as it could hold. The wet screen was allowed to stand for a few minutes to drain and its bottom edge afterwards touched with a piece of dry filter paper to remove the surplus amount of solution held by it. Readings were first taken with the screen wet. It was then dried by allowing it to stand in a current of hot air. To moisten it again when necessary, it was held in a steam bath, care being taken that too much moisture did not deposit on it. By this means the two conditions could be repeated at will and after the initial stages there was no necessity to touch the screens.

Method of Charging the Electroscopes.—As no high potential battery was available to charge the electroscopes, another method had to be employed, which would be as efficient and convenient.

Fig. 3 gives a diagram of the connections in the method finally adopted. The scheme is based on the principle of the Blake Static Machine. By this means high potentials could be obtained by using an accumulator giving only 4 volts. The accumulator drives a small sparking coil whose connections are shown in the diagram. One terminal of the secondary of the sparking coil is connected to the electrode *A*, which dips into a vertical glass tube containing a liquid of very high resistance; alcohol was used in this case but other high resistance liquids would answer the purpose equally well. Attached to the lower end of this vertical tube, is the other electrode *D*, which terminates in a point. Below this sharp point is a sphere of about half an inch radius which is connected to a variable high resistance *C*. The other terminal of this resistance is connected to the insulated charging rod of the electroscope. This second high resistance was a long piece of thread rendered conductive by lightly rubbing it with graphite, different lengths of which could be used by moving the slide *S*. The spark gap at *G* must be carefully adjusted, but once it is fixed, there is no need to readjust it, if dust is kept from depositing on the brass ball *B*. To guard against this, the tube and spark gap were enclosed in a wooden box. The electroscopes could be charged at any desired rate by adjusting the resistance *C*.

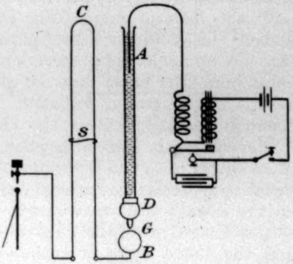

FIG. 3.

Method of Procedure.—The usual method was adopted to take readings; the deflection of the secondary electroscope was observed for a definite deflection of the primary electroscope. The characteristic radiations of the four elements, *Cu*, *Br*, *Ag* and *Sn*, were employed in the investigation ; in the case of the two salts chosen the softer radiations excited only the *L* electrons of the heaviest atoms, whilst the harder radiations called out both *K* and *L* electrons. With each characteristic radiation employed, a definite cycle of operations was carried out with each salt, both in the dry and in the wet state. This cycle consisted of four steps, in each of which the screens were arranged inside the chamber in a different order. The arrangements are shown in Fig. 2 (*b*), the arrows indicating the direction of propagation of the exciting radiation.

In (1) the two salt screens are innermost, and in this case both emergent and incident corpuscular radiation enter the chamber. In (2) the incident corpuscular radiation only enters the chamber. In (3) the two filter paper screens are innermost, so that no corpuscular radiation enters the chamber. It is assumed that pure filter paper does not give rise to an appreciable amount of corpuscular radiation. In (4) only emergent corpuscular radiation enters the chamber. The incident X-radiation suffers the same absorption in each case. These observations enable us to find the values of the incident and the emergent corpuscular radiations emitted by the screens. When the screens that were at the back of the chamber were interchanged for those that were previously at the front and the above cycle of operations repeated, the ratio of emergent to incident corpuscular radiation was found to be the same as that obtained before interchanging.

The observed ratio just found does not give the true ratio required, namely that of the number of emergent corpuscles to the number of incident corpuscles emitted by a layer of the substance so thin that the exciting radiation suffers no absorption in it. In the case we are investigating there is a correction to be made for the absorption of both the X-rays and the corpuscular rays in the screens and also the absorption of the X-rays in the layer of air between the two inner screens.

Fig. 4 represents diagramatically the conditions obtaining in the experiment. Let I be the intensity of the X-radiation at the inner face B of the first salt screen whose thickness is t_1, and let μ_1 be the absorption coefficient of the rays in this screen. The rays traverse the screen in the direction of the arrows. The intensity of the rays at a depth x from the face B is given by the relation

$$I=I_x \,.\, e^{-\mu_1 x}.$$

The number of corpuscles formed in a layer of thickness dx at a depth x in the screen

$$=\kappa I_x dx=\kappa I e^{+\mu_1 x}\,dx,$$

where κ is a constant.

If β be the absorption coefficient of the corpuscles in the screen, the number of corpuscles that reach the inside of the chamber from the layer dx

$$=\kappa I e^{+\mu_1 x} dx \,.\, e^{-\beta x},$$

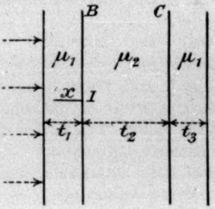

FIG. 4.

so that the total number of corpuscles entering chamber from the whole plate

$$=\kappa I\int_0^{t_1} e^{-(\beta-\mu_1)x}\,dx,$$

$$=\frac{\kappa I}{\beta-\mu_1},$$

since $e^{-\beta {}_1}=0$ because the screen behaves as a plate of infinite thickness to the corpuscular radiation.

Hence the observed ionisation produced by the emergent corpuscles is proportional to

$$\frac{\kappa I}{\beta-\mu_1}.$$

If t_2 be the thickness of the layer of air between the two screens, and μ_2 be the coefficient of absorption of the X-radiation in air, then the intensity of the X-radiation at C, the inner surface of the second screen, is $Ie^{-\mu_2 t_2}$. The intensity at depth x in the screen $=Ie^{-\mu_2 t_2-\mu_1 x}$. If r be the ratio of the number of corpuscles emitted in the direction in which the X-rays are travelling to that in the opposite direction from a thin layer dx, then the number of corpuscles that reach the chamber from the second screen is

$$\frac{\kappa I}{r} \,.\, e^{-\mu_2 t_2 - \mu_1 x} \,.\, e^{-\beta x}\, dx.$$

Total number of incident corpuscles from the whole screen

$$= \frac{\kappa I}{r} \,.\, e^{-\mu_2 t_2} \int_0^{t_3} e^{-(\beta+\mu_1)x}\, dx,$$

$$= \frac{\kappa I}{r} \,.\, e^{-\mu_2 t_2} \,.\, \frac{1}{\beta+\mu_1}, \quad \text{since } e^{-\beta t_3} = o,$$

which is proportional to the observed ionisation produced by the incident corpuscles.

Hence r, the true ratio required, $= r_0 \,.\, e^{-\mu_2 t_2} \,.\, \frac{\beta-\mu_1}{\beta+\mu_1}$, where r_0 is the observed ratio.

Barkla and Collier [*] and Owen [†] have determined the absorption coefficients of a number of homogeneous X-radiations in air, so that the values of μ_2 for the different radiations here employed are directly obtainable.

There are no direct data obtainable concerning the values β and μ_1. Approximate values of these quantities may, however, be obtained from existing data. Lenard found that the absorption of fast moving cathode particles in different substances is roughly proportional to the densities of those substances. Approximate values of the absorption of the cathode rays excited by the different X-radiation used, may, therefore, be calculated from the values of the absorption coefficients in air of the corpuscular radiations produced by these characteristic X-rays. These absorption coefficients have been experimentally determined by Beatty [‡] and by Sadler. [§]

The values of the absorption coefficients of the X-rays in the salts employed have been calculated from the values obtained by Barkla of the absorptions of the rays in different elements, making use of the relation found by Bragg and Pierce [||] between the atomic number of the absorber and atomic absorption coefficient. An approximate value is obtained for the molecular absorption coefficient of the salt for each

[*] Barkla and Collier, Phil Mag., Vol. XXIII., p. 987, 1912.

[†] Owen, Proc. Roy. Soc. A., Vol. LXXXVI., p. 426, 1912.

[‡] Beatty, Phil. Mag., Vol. XX., p. 324, 1910.

[§] Sadler, Phil. Mag., Vol. XXII., p. 447, 1911.

[||] Bragg and Pierce, Phil. Mag., Vol. XXVIII., p. 620, 1914.

of the rays employed, from which the ordinary absorption coefficient is immediately deduced. The values employed for μ_2, β and μ_1 are given in Table I., and the correcting factors calculated for each case. The tabulated values of the correcting factors strictly apply to the dry screens, but no appreciable error is introduced by assuming them to apply also to the case of the wet screens.

The final results are collected together in Table II. The figures show that the value of the ratio of emergent to incident corpuscular radiation in the case of the two salts investigated does not differ appreciably from the values obtained for this ratio by Cooksey and by Philpot in the case of gold and silver in the pure metallic state. There is an indication of an increase in the value of the ratio with increase in hardness of the exciting radiation; the variation is so slight, however, that the value obtained by taking the mean of all the values found for the ratio with the different exciting radiations, may be taken as representing a very approximate value of the ratio in each case. This mean value is practically the same for the

TABLE I.

Exciting Radiation.	μ_2.	Potassium Bromide.			Silver Nitrate.		
		μ_1.	$\beta\times10^{-4}$.	$\frac{\beta-\mu_1}{\beta+\mu_1}\cdot e^{-\mu_2 t_2}$.	μ_1.	$\beta\times10^{-4}$.	$\frac{\beta-\mu_1}{\beta+\mu_1}\cdot e^{-\mu_2 t_2}$.
Cu	0·0109	310	11·1	0·97	607	17·4	0·97
Br	0·0039	93	5·6	0·99	212	8·8	0·99
Ag	0·0008	83	1·9	0·99	38	3·0	0·99
Sn	0·0004	14	1·4	1·00	45	2·2	1·00

TABLE II.

Exciting Radiation.	Potassium Bromide.				Silver Nitrate.			
	Wet.		Dry.		Wet.		Dry.	
	Observed ratio.	Corrected ratio.	Observed ratio.	Corrected ratio.	Observed ratio.	Corrected ratio.	Observed ratio.	Corrected ratio.
Cu	1·18	1·14	1·20	1·16	1·20	1·16	1·19	1·15
Br	1·15	1·14	1·15	1·14	1·19	1·18	1·15	1·14
Ag	1·19	1·18	1·20	1·19	1·18	1·17	1·20	1·19
Sn	1·19	1·19	1·18	1·18	1·20	1·20	1·22	1·22
Mean Values		1·16		1·17		1·18		1·17

two salts, the final mean value being 1·17, which is in good agreement with the value of the ratio found by the two above mentioned observers. Further there does not seem to be a difference in the ratio whether the salt is in the amorphous or in the crystalline state.

These results would indicate that the asymmetry observed in the case of corpuscular radiation produced by X-rays of wave-length ranging from the characteristic radiation of copper to that of tin, is practically constant and independent of the nature of the screen which emits the corpuscles, both as regards the substance of the screen and the state of that substance.

Summary.

1. The ratio of emergent to incident corpuscular radiation in the case of the two salts, potassium bromide and silver nitrate, has been investigated, when the exciting X-radiations were the characteristic radiations of copper, bromine, silver and tin.

2. The ratio has the same value whether the salt is in the wet or in the dry state.

3. The value of the ratio was found to be approximately the same for each of the two salts, and is equal to 1·17. This is approximately the same figure as that found by other observers in the case of the metals, gold and silver.

It gives me pleasure to express my appreciation of the kindly interest which Prof. A. W. Porter, F.R.S., has taken in the work.

My best thanks are due to Mr. G. G. Blake for helping me throughout the investigation.

ABSTRACT.

1. The ratio of emergent to incident corpuscular radiation in the case of the two salts, potassium bromide and silver nitrate, has been investigated, when the exciting X-radiations were the characteristic radiations of copper, bromine, silver and tin.

2. The ratio has the same value whether the salt is in the wet or in the dry state.

3. The value of the ratio was found to be approximately the same for each two of the salts, and is equal to 1·17. This is approximately the same figure as that found by other observers in the case of the metals, gold and silver.

DISCUSSION.

Dr. D. Owen said the Author's experiments bore the stamp of accuracy, and appeared to give a decisive answer to the problem proposed. On general principles, however, the observed preponderance in the number of secondary β-rays emitted on the emergent side (when X-rays fall upon a thin plate) appeared a natural result to expect. For the X-ray pulse carries not only energy, but also momentum; and the electrons within the atoms of the target or radiator experience not only transverse electric force, but also a forward impulse. On the view of a continuous wave-front, this imparted momentum may be insufficient to jerk an electron out of the atom, just as on that view the energy of the pulse has been calculated to be insufficient to produce the observed ionisation due to X-rays. On the view, however, of the existence of quanta—*i.e.*, intense condensations of activity in the wave-front—the forward momentum of the beam may be impressed like a series of hammer blows on the atoms of the target. A simple calculation shows that appreciable effects in this respect may well occur—*e.g.*, in the case of radiation of the intensity of full sunlight the momentum communicated would, if wholly spent in ejecting electrons on the emergent side, be sufficient to generate 10^{15} electrons per square centimetre per second, each with a velocity of 10^9 cm. per second.

Dr. H. S. **Allen** said that if he understood the Paper rightly, the result that Mr. Owen has found—viz., that the ratio of emergent to incident corpuscular radiation is the same in the amorphous as in the crystalline state, renders doubtful the explanation of asymmetry put forward by H. A. Wilson. In his opinion an explanation of a more fundamental character was required than that which attributes the asymmetry of the scattered radiation to the difference between the behaviour of crystalline and amorphous material. In this connection a recent Paper by A. H. Compton ("Journ." Wash. Acad. Sci., January 4, 1918) is of considerable interest. He assumes that the electron is in the form of a spherical shell, each part of which can scatter independently, and may be capable of rotational motion. He shows that it is then possible to explain not only the asymmetry of the scattered rays, but also the diminution of scattering with decrease of wave length. Since the mass of an electron cannot be accounted for on the basis of a uniform distribution of electricity over the surface of a sphere, Mr. Compton suggests that the true shape of the electron may be that of a ring, having an effective radius many times greater than that ordinarily accepted. Mr. Compton's estimate of the radius is $2{\cdot}3 \times 10^{-10}$ cm., but if some recent measurements by Sir Ernest Rutherford are used in the calculation, this estimate must be reduced to about one-tenth of the value stated ("Nature," Vol. C., p. 510, 1918). If this hypothesis of a ring electron be accepted, the electron may act as a small magnet, as suggested by A. L. Parson, and this explains Forman's effect of magnetisation of iron upon its absorption coefficient.

Mr. T. **Smith** thought the initial equation with which the Paper started was somewhat extraordinary. It was difficult to see what physical considerations gave rise to the factor $\cos \frac{1}{2}\theta$ in the denominator of an equation which he presumed was intended to apply from 0 deg. to 180 deg.

The **Author**, in reply, said that if the asymmetry is to be attributed to the pressure of radiation, one would expect that the value of the ratio obtained for the asymmetry would vary with the intensity of the exciting X-radiation. It has been shown, however, by Philpot that this is not the case, the value of the ratio remaining the same for radiations of the same wave-length whether a beam of X-rays direct from a bulb or a beam of characteristic rays from a metal plate, were employed to excite the corpuscular radiation; the intensity of the radiation would be much greater in the former than in the latter case. The results of the present Paper show that the explanation put forward by Wilson to explain the asymmetry in the case of X-radiation does not appear to be adequate to explain the asymmetry in the case of corpuscular radiation. He agreed with Dr. Allen that an explanation of a more fundamental character is necessary to account for this phenomenon.

XII. *On "Air Standard" Internal Combustion Engine Cycles and their Efficiencies. By* CHARLES H. LEES, *D.Sc., F.R.S.*

RECEIVED FEBRUARY 18, 1918.

1. In accordance with the recommendations of the Committee of the Institution of Civil Engineers on the Standards of Efficiency of Internal Combustion Engines* it has become the practice to compare the thermal efficiency of any internal combustion engine with that of an ideal standard engine taking in and giving out its heat in as nearly as possible the same way as the actual engine, using as working substance a perfect gas, and subject to no losses due to friction or to conduction or radiation of heat from the working substance. Such an ideal engine using air, which for the purpose in view may be taken as a perfect gas, the Committee calls an "Air Standard" engine.

In such an ideal engine the heat produced by the explosion of the charge is supposed to be communicated to the working substance without changing the physical properties of that substance, the expansion which follows is taken as strictly adiabatic, the exhaustion and re-charge of the cylinder are supposed to occur instantaneously at the end of the working stroke, or the working substance is supposed to remain the same but to have sufficient heat taken from it to reduce its temperature and pressure to those existing at the beginning of the compression stroke, which in turn is supposed to be strictly adiabatic.

2. Two of the processes which take place during the cycle of operations are thus seen to be adiabatic, while the other two processes by which the working substance is brought from one adiabatic to the other may be of any type, either the same or different, so long as they are not adiabatic. Up to the present time the only other processes which have been considered in the discussions of the thermal efficiencies of Air Standard internal combustion engines are those at :—

(*a*) constant temperature ;
(*b*) constant pressure ;
(*c*) constant volume.

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* Proc. Inst. Civil Eng., Vol. CLXII., p. 507 (1905) and Vol. CLXIII., p. 241 (1906). The Committee consisted of Profs. Ashcroft, Callendar and Dalby, Sir Dugald Clerk, Mr. Hayward, Sir Alexander Kennedy, Capt. Sankey and Mr. Wilson.

The first give with the two adiabatic strokes the Carnot cycle, the second give a cycle partially followed in the Diesel engine, while the third give the Otto and the Clerk cycles.

It is pointed out in the Report of the Committee on the Standards of Efficiency that for each of these cycles the expression for the efficiency is the same,* viz. :—

$$\text{efficiency}=1-(1/r)^{\gamma-1},$$

where γ is the ratio of the specific heats of the working substance at constant pressure and at constant volume respectively and r is the ratio of the volume of the substance before to that after the compression stroke a ratio, known as the "compression ratio."

3. It is one of the objects of the present Paper to show that the efficiencies of a much more extensive class of cases than the three named in the Committee's Report are expressed by the same function of the "compression ratio."

We shall assume that the adiabatic curve of the working substance is given by

$$pv^{\gamma}=\text{constant},$$

where p is the pressure and v the volume, and that the constant has the value C at the expansion and c at the compression stroke. We shall take the curves for the periods of absorption and rejection of heat which complete the cycle to be of the form

$$pv^{\alpha}=\text{constant},$$

where α is a constant not identical with γ and the constant on the right of the equation has the value A for the explosion and a for the exhaust curve. Writing the values of the pressure and volume at the end of the compression $p_0\,v_0$, at the end of the heat absorption $p_1\,v_1$, at the end of the adiabatic power stroke $p_2\,v_2$, and after heat rejection $p_3\,v_3$, we have (Fig. 1) :—

$$p_0v_0^{\alpha}=p_1v_1^{\alpha},\quad p_1v_1^{\gamma}=p_2v_2^{\gamma},\quad p_2v_2^{\alpha}=p_3v_3^{\alpha},\quad p_3v_3^{\gamma}=p_0v_0^{\gamma}.$$

Multiplying together these equations we get

$$p_0p_1p_2p_3v_0^{\alpha}v_1^{\gamma}v_2^{\alpha}v_3^{\gamma}=p_1p_2p_3p_0v_1^{\alpha}v_2^{\gamma}v_3^{\alpha}v_0^{\gamma},$$

or $$v_1v_3=v_0v_2,\quad \text{that is,}\quad v_3/v_0=v_2/v_1,$$

or the compression ratio along the compression adiabatic is equal to the expansion ratio along the expansion adiabatic.

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* Proc. Inst. Civil Eng., Vol. CLXII., p. 325 (1905). The author of this simple expression for the efficiencies in the three cases is not indicated in the report, but according to Sir Dugald Clerk, "The Gas, Petrol and Oil Engine," new edit., Vol. I., note p. 83, it was first pointed out to the Committee by Prof. Callendar.

Writing the equation $v_1/v_0=v_2/v_3$ it is seen that equality of ratio holds also along the two a curves.*

4. The efficiency of the cycle is equal to $1-H_{23}/H_{01}$ where H_{23} is the heat given up by the working substance in the path from p_2v_2 to p_3v_3, and H_{01} the heat taken in by it in the path from p_0v_0 to p_1v_1. In each case the heat is measured in work units by the area enclosed by the path, and those parts of the two adiabatics through the ends of the path which lie to the right of the path.

The area included between the ordinate of the point p_3v_3, the axis of volume and that part of the adiabatic $pv^\gamma=c$ through the point which lies to the right of it, is $\dfrac{p_3v_3}{\gamma-1}$.

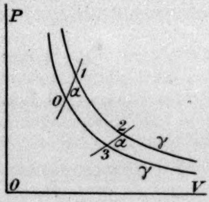

FIG. 1.

Curve 1-2 adiabatic $pv^\gamma=C$. 0-1 $pv^a=A$.
" 0-3 " $pv^\gamma=c'$. 3-2 $pv^a=a$.

That between the ordinate of the point p_3v_3, the curve $pv^a=a$ between p_3v_3 and p_2v_2 and the ordinate of the latter point is $\dfrac{(p_2v_2-p_3v_3)}{a-1}$, which reduces to $p_3v_3\log(v_2/v_3)$, if $a=1$.

In the general case the area between the curve $pv^a=a$ through the points p_3v_3, p_2v_2, and those portions of the adiabatics through the same points lying to the right of them is therefore

$$=(p_2v_2-p_3v_3)\left(\frac{1}{a-1}-\frac{1}{\gamma-1}\right)$$

Hence this expression is the H_{23} of the efficiency formula. It reduces to $p_3v_3\log(v_2/v_3)$ if $a=1$.

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* These relations, as well as the corresponding ones connecting the pressures, are readily seen from a diagram with $\log v$ and $\log p$ as coordinates.

In the same way it is seen that

$$H_{01}=(p_1v_1-p_0v_0)\left(\frac{1}{\alpha-1}-\frac{1}{\gamma-1}\right),$$

which reduces to $p_0v_0\log(v_1/v_0)$ if $\alpha=1$.
Hence the efficiency of the cycle becomes

$$1-\frac{(p_2v_2-p_3v_3)}{(p_1v_1-p_0v_0)}.$$

That is since $p_2v_2^\alpha=p_3v_3^\alpha$, $p_1v_1^\alpha=p_0v_0^\alpha$, and $v_2/v_3=v_1/v_0$

$$1-p_3v_3/p_0v_0.$$

Or finally since $p_3v_3^\gamma=p_0v_0^\gamma$, the efficiency may be written

$$1-(v_0/v_3)^{\gamma-1} \quad \text{or} \quad 1-\left(\frac{1}{r}\right)^{\gamma-1} \quad \text{where} \quad r=\frac{v_3}{v_0}.$$

The special value $\alpha=1$ leads to the same expression.

Thus a cycle composed of two adiabatics and two curves of the form pv^α=constant, where α is a constant which may have any value, positive or negative, so long as it is the same for the two curves, has an efficiency$=1-(1/r)^{\gamma-1}$, where r is the compression ratio measured along either adiabatic.

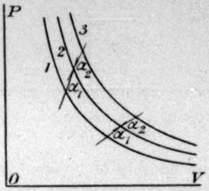

FIG. 2.

Curves 1, 2, 3 adiabatics pv^γ=constant.
α_1, α_2, values of α for curves pv^α=constant.

This general law includes the three cases for which the expression was previously known to hold, *i.e.*, for constant temperature ($\alpha=-1$), constant pressure ($\alpha=0$), and constant volume ($\alpha=\infty$).

5. If a third adiabatic is drawn and we designate the three in order of distance from the origin the first, second and third adiabatic respectively (Fig. 2) we may take two points on the first adiabatic for which the compression ratio r has a given value, and can draw through each of these points a curve of

the class pv^a=constant with the special value a_1 for a. These curves will cut the second adiabatic in points which have the same compression ratio, r, as the first pair of points. Through each of the points on the second adiabatic we can draw a second curve pv^a=constant, with $a=a_2$. These will cut the third adiabatic in points for which the compression ratio still has the same value r. The process may be continued to, say, the nth adiabatic. Between the first and the nth adiabatic we now have a cycle whose explosion and exhaust curves are made up of portions of curves of the type pv^a=constant with the values of a for the successive portions of curve chosen in any way we desire. The efficiency of the cycle is still given by $1-(1/r)^{(\gamma-1)}$ where r is the compression ratio along any of the adiabatics. The number of intermediate adiabatics and the value of a between each pair may be so chosen that any prescribed curve for either the explosion or for the exhaust may be followed as closely as we desire.

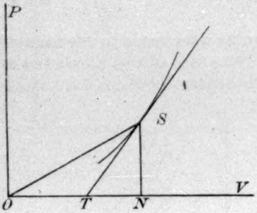

FIG. 3.—DETERMINATION OF a FOR CURVE AND COTSTRUCTION OF CURVE WITH GIVEN a.

If, for instance, $p=f(v)$ is the curve with which pv^a=constant is to be made to coincide in the neighbourhood of the point p_0v_0, we make $pv^a=p_0v_0^a$ to secure that the a curve passes through the point p_0v_0 and the slope upwards $-ap_0/v_0=f'(v_0)=\tan\theta_0'$, where θ_0' is the angle which the tangent to the given curve at the point p_0v_0 makes with the axis of volumes. This gives $a=-\tan\theta_0'/\tan\theta_0$, where θ_0 is the angle which the radius vector to the point p_0v_0 makes with the axis of volumes (Fig. 3) or if N is the foot of the perpendicular from the point S on to the axis of volumes and T is the point in which the tangent to the curve at S cuts that axis, $a=-ON/TN=ON/NT$, where the subtangent NT is regarded as positive if T lies on the further side of N from O.

If a is given the curve pv^a=constant through a given point may be drawn in steps by the same construction.

If the curve which has not been used in selecting the values of a be now drawn with the values of a thus determined we have a cycle which still has the efficiency $1-\left(\frac{1}{r}\right)^{(\gamma-1)}$.

Thus an "air standard" cycle composed of two adiabatics pv^γ=constant, and either an explosion curve or an exhaust curve of any prescribed form, the curve not prescribed being drawn with the values of a determined for the prescribed curve, will have its efficiency given by $1-(1/r)^{\gamma-1)}$, where r is the compression ratio measured along either adiabatic.

6. In applying this theorem to any "air standard" cycle composed of two adiabatics and two other curves crossing them which may for convenience be called the explosion and exhaust curves respectively, it is best to select values of a which will reproduce as closely as it is desired the explosion curve $ABCD$ (Fig. 4), and with these values of a to draw the exhaust curve $A'B'C'D'$ to correspond. If this is done the efficiency of the actual cycle can be readily determined. For the efficiency of the cycle $ABCDD'C'B'A'$ with corresponding explosion and exhaust curves is given by $1-(1/r)^{\gamma-1}$, where r is the ratio of compression measured along either adiabatic, and γ is the exponent for the adiabatic. This expression is the ratio of the area enclosed by the cycle $ABCDD'C'B'A'$

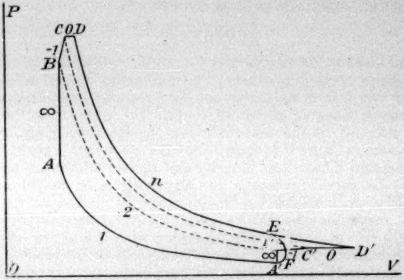

FIG. 4.

Curves 1, 2, ... n adiabatics pv^a = constants. $ABDEF$ actual cycle. $ABDD'C'A'$ cycle with corresponding explosion and exhaust curves. ∞, -1 and O are the values of a for the lines against which they are written.

to the area which represents in work units the heat taken in by the working substance along the explosion curve *ABCD*. In the same manner the efficiency of the actual cycle *ABCDEFA′* is the ratio of the area enclosed by that cycle to the heat taken in along the explosion curve. Hence as the two explosion curves have been made identical, as nearly as desired, the efficiency of the actual cycle is

$$\left\{1-\left(\frac{1}{r}\right)^{\overline{\gamma-1}}\right\}\frac{\text{area of actual cycle } ADEF}{\text{area of cycle } ADD'B' \text{ with exhaust curve with same } a \text{ values as explosion curve.}}$$

For many purposes a sufficiently close approximation to the actual explosion curve may be secured by taking in succession a constant volume line, a line through the zero of volume and pressure, and a constant pressure line as at *ABCD* (Fig. 4). These lines are easily drawn and can be so selected that the area enclosed by them and the actual curve is as much on one side of them as on the other.

ABSTRACT.

It is well known that the efficiency of an air standard internal combustion engine working through a cycle bounded by two adiabatics, and either two isothermals, two constant volume lines or two constant pressure lines is given by $1-(1/r)^{\gamma-1}$ where r is the compression ratio and γ is the ratio of the two specific heats of air.

In the present Paper it is shown that the efficiency is given by the same expression if the cycle is composed of two adiabatics and two curves $pv^a=A$, $pv^a=a$, where a has any positive or negative value and A and a are constants. Since a may be chosen so that any explosion curve may be followed as closely as desired by short lengths of a curves, a cycle can be drawn with the above efficiency and any prescribed explosion curve. The ratio of the efficiency of a cycle with prescribed explosion and exhaust curves to that of the cycle so drawn is shown to be the ratio of the two areas on the indicator diagram. The thermal efficiency of a cycle with prescribed explosion and exhaust curves is therefore readily found.

DISCUSSION.

Dr. D. Owen asked what the ratio of the efficiencies of the actual and theoretical cycles was in practice.

Dr. H. S. Allen asked if it was not possible to generalise the result established in the Paper, and to say that the expression for the efficiency held when the explosion curve is represented by $f(p, v)=\text{constant}$, where $f(p, v)$ is any function of p and v, provided the equation for the exhaust curve is suitably chosen.

Prof. Lees, in reply, said the ratio of the actual cycle to the other was usually 0·9 to 0·95. Dr. Allen's suggestion was quite correct. He had aimed, however, at expressing the result in a form suitable for graphical calculation.

XIII. *Cohesion (Fourth Paper).* *By* HERBERT CHATLEY, *D.Sc. (Lond.).*

RECEIVED FEBRUARY 25, 1918.

THE aim of the present Paper is to consider the value of molecular force as indicated by Van der Waals' gas formula (particularly at the critical state where the liquid and gaseous states merge), and to relate the results to the previous inquiry.

The results of the investigation may be summarised as follows :—

1. Van der Waals' formula implies that molecular attraction varies inversely as the fourth power of the molecular interval.* Incidentally it may be mentioned that it also indicates the repulsion as varying inversely as the molecular interval, except for very small intervals, where it varies as the square of the interval divided by the difference between the cubes of the interval and of the interval at absolute zero, but also depends on the thermal energy involved.

2. The approximate accuracy of Van der Waals' formula at the critical state provides a further disproof of Kelvin's Newtonian theory of molecular attraction.

3. The author's formula,†

$$t_2 = Gm^2/d^{2+4\frac{d_0}{d}},$$

is apparently not irreconcilable with the actual values of the molecular attraction at the critical state.

4. Certain modifications must be introduced in the Van der Waals' formula to make it agree exactly with the facts.

1. *Deductions from Van der Waals' Formula.*

It is well known that this formula is arrived at as follows :—

$$\left\{\begin{matrix}\text{Ideal unrestricted}\\ \text{kinetic gas pressure}\end{matrix}\right\}\left\{\begin{matrix}\text{Clear volume in}\\ \text{one mol of gas}\end{matrix}\right\}=RT,$$

or

$$\left\{\begin{matrix}\text{Actual gas pressure + pressure}\\ \text{due to molecular centripetal}\\ \text{attraction of the whole mass}\end{matrix}\right\}\left\{\begin{matrix}\text{Actual molar}\\ \text{volume} - \text{net volume}\\ \text{of molecules}\end{matrix}\right\}=RT,$$

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* The term interval here means the distance from centre to centre.

† *Vide* "Cohesion," third Paper.

which was written by Van der Waals as

$$\left(p+\frac{a}{v^2}\right)(v-b)=RT.$$

This is a cubic equation in terms of v, and at the critical state the three roots are equal, and we have the following relations

$$a=27b^2p_c=3p_cv_c^2=\frac{27RT_cb}{8}=\frac{9}{8}RT_cv_c=\frac{27R^2T_c^2}{64p_c},$$

$$b=\frac{v_c}{3}=\frac{RT_c}{8p_c},$$

and also

$$p_cv_c=\tfrac{3}{8}\,.\,RT_c,^*$$

and

$$\frac{a}{v_c^2}=3p_c\,;\quad p_c'=p_c+\frac{a}{v_c^2}=\frac{RT_c}{v_c-b}=4p_c,\text{ so that }\frac{p_c'}{p_c}=4\text{ and }\frac{p_c'}{\frac{a}{v_c^2}}=\frac{4}{3}.$$

Assuming no expansion of the actual molecules, $b=v_0$, the molar volume at absolute zero, and so we find that at the critical state the molecular interval

$$d_c=\sqrt[3]{3}\,.\,d_0=1{\cdot}442d_0,$$

and, assuming cubic spacing, $d_0=\sqrt[3]{\frac{b}{N}}$, where N is Avogadro's number (molecules per gram-molecule).

The radius of the molar sphere whose volume is v, is

$$r=\sqrt[3]{\frac{3v}{4\pi}},$$

and the density $D=M/v$.

The number of molecules per square centimetre

$$=\left(\frac{N}{v}\right)^{\frac{2}{3}}=\frac{1}{d^2}=n^2.$$

The attraction on a single molecule at the surface of the molar sphere due to all the N molecules in it, is then, according to Van der Waals,

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* This formula, assuming Van der Waals' equation is true, expresses the relation for the production of critical volume at high temperatures and pressure. Thus, for hydrogen, $T_c=2{\cdot}354\times10^{-6}p_c$, where p_c is in dynes per square centimetre. If p_c is 10^{10}, $T_c=23{,}540°$ *abs.*, or, say, 23,250°C. The Bridgman solar and planetary liquids doubtless occur under some such pressures and temperatures.

$$t=\frac{\frac{a}{v^2}}{n^2}=aN^{-\frac{2}{3}}v^{-\frac{4}{3}};$$

and, since the molecular interval $d=\sqrt[3]{\frac{v}{N}}$,

$$t=aN^{-2}d^{-4}.$$

Van der Waals' equation, therefore, implies that the attraction of the whole gram molecule (on one molecule) varies inversely as the fourth power of the molecular interval.

For CO_2, the value of a as computed from p_c and T_c is $3{\cdot}6\times10^{12}$, so that if d is 10^{-7},

$$t=\frac{3{\cdot}6\times10^{12}}{36\times10^{46}\times10^{-28}}=10^{-7} \text{ dynes}.$$

It must be observed that this is the attraction due to N molecules acting on one superficial molecule, and not that between one pair of molecules. All except a few, however, are very remote from the one considered, and therefore contribute only slightly to the total force. The effective distance of the whole mass when the law of attraction is *not* the inverse square is *not* the radius of the sphere.

2. *Comparison with the Newtonian Law.*

If the Newtonian law were applicable, as Kelvin thought, then we should have (M=molecular weight)

$$t_1=\frac{GMn^2m}{r^2}=\frac{GM^2}{N^{\frac{1}{3}}\left(\frac{3v}{4\pi}\right)^{\frac{2}{3}}}, \text{ per square centimetre.}$$

This disagrees dimensionally with Van der Waals' form. Using the actual critical value of v for CO_2

$$t_1=1{\cdot}844\times10^{-13} \text{ dynes per square centimetre,}$$

or, $2{\cdot}624\times10^{-29}$ dynes per superficial molecule, whereas the actual molecular attraction as computed from the critical values of volume, pressure and temperature is $3{\cdot}784\times10^{8}$ dynes per square centimetre, or $8{\cdot}744\times10^{-7}$ dynes per molecule, making the ratio

$$\frac{\text{molecular attraction}}{\text{Newtonian attraction}}=3{\cdot}33\times10^{22}.$$

This may be compared with the ratio of 10^{30}, which is given in "Cohesion" (second Paper) for the solid state. Kelvin's hypothesis thus appears to be decisively controverted. Further values of the attractions are given in the attached table.

The apparent discrepancy of considering here the attraction of many molecules upon one, whereas in the analysis of the solid state only single pairs were taken is partially explained by the fact that in the gas or even in the liquid there are repeated collisions, successively bringing many molecules into close proximity with the one in question.

3. *Comparison with the Author's Formula.*

It seems impossible to integrate the attraction of a sphere when the index of the molecular interval varies, but quantitatively the values given by the author's formula for small distances appear to be not impossible. The approximate accuracy of Van der Waals' formula for large values of v, in spite of the fact that the author's formula then indicates the Newtonian condition, is possibly also explicable by the state of repeated collision, although at or near the critical condition the collisions will mostly occur between the same pairs.*

At the critical state a comparison of the actual molecular attraction with that obtained by the author's formula shows that the attraction of the N molecules (whose centre is at a distance r from the superficial molecule) is from 10^8 to 10^{21} times that of a single molecule at the molecular interval. Since N is $6{\cdot}06 \times 10^{23}$, this shows a fair agreement, but the question needs to be further investigated.

The following formula is used to compute the actual attractions :—

$$p_c(1+A) \, . \, v_c(1-B)=RT_c,$$

and experimentally $p_cv_c=kRT_c$;

$$A=\frac{1}{(1-B)k}-1\,;$$

the attraction per superficial molecule$=Ap_cd_c^2$.

At absolute zero there is very good agreement.

4. *Corrections to Van der Waals' Formula.*

The discrepancies of the Van der Waals' formula at the critical state are as follows :—

(*a*) The irreducible volume is too large.
(*b*) The critical volume is too large.
(*c*) The ratio $p_cv\,/RT_c$ is too large.

* See footnote on next page.

The quantities a and b must, therefore, be reduced slightly, but the differences vary with different substances.

The index of v in the attraction term is certainly incorrect for very large values of v, although as mentioned above, the continuous series of temporary close contacts in collision may mask this error.*

The table which follows shows the principal results which can be obtained along these lines. It is hoped to proceed further and connect the conditions with those in the solid state. However this may be, it seems clear that Van der Waals' hypothesis throws important light on the question of molecular astronomy.

TABLE OE DATA.

Substance.	H_2.	H_2O.	CO_2.	SO_2.	$SnCl_4$.
Molecular weight…………	2·0155	18·0155	44·00	64·10	260·3
Critical pressure ; Atmospheres.	13·4	217·5	72·9	77·70	36·95
Critical temp. : Absolute………	31·0	647·0	304·1	430·3	591·8
Critical volume : v_c cubic cm. …	60·45	45·00	98·22	125·00	351·00
Ditto from V. der W.s' equat.	72·10	91·50	128·30	170·30	492·60
a from p_c and T_c …………	$0{\cdot}21\times10^{12}$	$5{\cdot}537\times10^{12}$	$3{\cdot}651\times10^{12}$	$6{\cdot}853\times10^{12}$	$13{\cdot}436\times10^{12}$
b from p_c and T_c …………	24·03	30·49	42·77	56·7	164·2
b_1=one-third critical volume …	20·15	15·00	32·74	41·67	117·0
b_2=volume at abs. zero ………	22·4	18·02	$<29{\cdot}33$	$<44{\cdot}83$	$<114{\cdot}1$
Critical density, D_c…………	0·033	0·4	0·448	0·513	0·7419
Density at abs. zero, D_0 ………	0·09	1·0	1·5	$>1{\cdot}43$	$>2{\cdot}28$
$D_c/D_0=v_0/v_c=B$ …………	0·36	0·4	0·299	$<0{\cdot}263$	$<0{\cdot}232$
$p_cv_c/RT_c=K$ …………	0·3217	0·1845	0·2869	0·2749	0·2670
A, from K and B …………	3·856	8·036	3·980	3·935	3·876
$p_c \div p'_c$ …………	0·2059	0·1107	0·2008	0·2027	0·2051
$p'_c \div Ap_c$ …………	1·26	1·125	1·251	1·254	1·258
$\sqrt[3]{1/B}=d_c/d_0$ …………	1·397	1·257	1·489	1·561	1·628
$d_c=\sqrt[3]{v_c/N}$ …………	$3{\cdot}331\times10^{-8}$	$4{\cdot}203\times10^{-8}$	$5{\cdot}451\times10^{-8}$	$5{\cdot}907\times10^{-8}$	$8{\cdot}835\times10^{-8}$
d_0 …………	$1{\cdot}673\times10^{-8}$	$3{\cdot}344\times10^{-8}$	$3{\cdot}660\times10^{-8}$	$3{\cdot}784\times10^{-}$	$5{\cdot}121\times10^{-8}$
$m=M/N$, gr. …………	$3{\cdot}326\times10^{-24}$	$2{\cdot}973\times10^{-23}$	$7{\cdot}259\times10^{-23}$	$1{\cdot}058\times10^{-22}$	$4{\cdot}294\times10^{-22}$
Gm^2 dynes …………	$7{\cdot}30\times10^{-55}$	$5{\cdot}833\times10^{-53}$	$3{\cdot}557\times10^{-52}$	$7{\cdot}387\times10^{-52}$	$1{\cdot}53\times10^{-50}$
GNm^2 dynes …………	$4{\cdot}415\times10^{-31}$	$3{\cdot}536\times10^{-29}$	$2{\cdot}156\times10^{-28}$	$4{\cdot}417\times10^{-28}$	$9{\cdot}27\times10^{-27}$
$r_c=\sqrt[3]{3v_c/4\pi}$ cm. …………	2·453	2·207	2·866	3·101	4·376
$F_2=GNm^2/r^2$ dynes …………	$7{\cdot}336\times10^{-32}$	$7{\cdot}259\times10^{-30}$	$2{\cdot}624\times10^{-29}$	$4{\cdot}593\times10^{-29}$	$4{\cdot}841\times10^{-28}$
$F_1=Ap_c/n_c^2=Ap_cd_c^2$ dynes ….	$1{\cdot}127\times10^{-8}$	$3{\cdot}130\times10^{-6}$	$8{\cdot}744\times10^{-7}$	$1{\cdot}079\times10^{-6}$	$1{\cdot}009\times10^{-6}$
$F_1 \div F_2$ …………	$1{\cdot}535\times10^{24}$	$4{\cdot}311\times10^{23}$	$3{\cdot}332\times10^{22}$	$2{\cdot}35\times10^{22}$	$2{\cdot}084\times10^{21}$
$\gamma=2+(4d_0/d_c)$ …………	4·869	5·183	4·692	4·563	4·458
$F_3=Gm^2/d_c\gamma$ dynes…………	$1{\cdot}299\times10^{-29}$	$9{\cdot}961\times10^{-15}$	$4{\cdot}280\times10^{-18}$	$7{\cdot}122\times10^{-19}$	$5{\cdot}538\times10^{-19}$
$\nu_1=F_1/F_3$ …………	$8{\cdot}672\times10^{21}$	$3{\cdot}142\times10^{8}$	$2{\cdot}043\times10^{11}$	$1{\cdot}516\times10^{12}$	$1{\cdot}822\times10^{12}$
ν_1/N…………	$1{\cdot}430\times10^{-2}$	$5{\cdot}183\times10^{-16}$	$3{\cdot}370\times10^{-13}$	$2{\cdot}500\times10^{-12}$	$3{\cdot}004\times10^{-12}$
$F_4=a/N^2d_0^4$ dynes …………	$4{\cdot}645\times10^{-7}$	$1{\cdot}206\times10^{-5}$	$5{\cdot}537\times10^{-6}$	$9{\cdot}093\times10^{-6}$	$5{\cdot}315\times10^{-6}$
$F_5=Gm^2/d_0^6$ dynes …………	$5{\cdot}35\times10^{-10}$	$4{\cdot}176\times10^{-8}$	$1{\cdot}122\times10^{-7}$	$2{\cdot}517\times10^{-7}$	$8{\cdot}285\times10^{-7}$
$\nu_2=F_4/F_5$ …………	868	288·8	49·33	10·01	6·415

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* A further discrepancy, which is here very serious, is that the attraction on the inner molecules effectively reduces their kinetic pressure on the outer ones, and so has the effect of apparently increasing the attraction on the outer ones.

ABSTRACT.

The Paper is the fourth of a series dealing with the subject of Cohesion. The aim of the present Paper is to consider the value of molecular force as indicated by Van der Waals gas formula (particularly at the critical state where the liquid and gaseous states merge), and to relate the results to the previous enquiry.

DISCUSSION.

Dr. H. S. Allen (communicated): In this Paper Prof. Chatley has pointed out a number of interesting relations depending on the equation of Van der Waals. It is more than doubtful whether any attempt to found a theory on purely central attractions and repulsions varying as some power of the distance can prove adequate to explain the facts. It is certain that the law of attraction at molecular distances is not that of the inverse square of the distance. In a lengthy series of Papers in the "Philosophical Magazine," Sutherland,* whose work has not received adequate recognition, has discussed the law of the inverse fourth power, and has pointed out the relation which it bears to the characteristic equations of Van der Waals and others. A useful summary of work on the subject is given by W. C. McC. Lewis, in his "System of Physical Chemistry" (Longmans, 1916). It may be mentioned that Lewis† has examined the connection between the internal pressure or cohesion, π, in the equation

$$(p+\quad)(v-b)=RT,$$

and the dielectric capacity and permeability of a liquid. He has shown that the Obach Walden relation regarding the proportionality between the internal pressure and the dielectric constant follows from the hypothesis that molecular attraction is electromagnetic, not electrostatic in nature. That cohesion arises from the action of electric or electro-magnetic forces may be inferred from optical experiments. The Lorentz-Fitzgerald hypothesis explains the negative result of the Michelson-Morley experiment by a contraction of the material framework of the apparatus in the direction of its motion through the aether. Such a contraction may be predicted from the standpoint of electromagnetic theory. There is, therefore, a strong presumption that "the forces of cohesion between the particles, which give a solid its rigidity, are electrical forces."‡

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* Sutherland, "Phil. Mag.," Vol. 24, p. 113, p. 168, 1887; Vol. 35, p. 211, 1893.

† Lewis. "Phil. Mag.," Vol. 28, p. 104, 1914.

‡ Eddington, "Nature," Vol. 101, p 15, 1918.

XIV. *Notes on the Pulfrich Refractometer.* By J. GUILD, *A.R.C.Sc., D.I.C., F.R.A.S.* (*from the National Physical Laboratory*).

RECEIVED MARCH 8, 1918.

§ 1. INTRODUCTION.

ALMOST all the commercial testing of optical glass is carried out on the instrument known as the Pulfrich Refractometer, the general principles of which are sufficiently well known to render a description of the usual type of instrument superfluous. Considerable experience in the use of this refractometer has led the author to form a very high opinion of its potentialities, but a very much poorer one of the performance of the actual instrument as turned out by Messrs. Zeiss—until recently the only makers. This is in large part due to the fact that work of the accuracy now attempted with the instrument was probably not contemplated by the designer, who most likely had in view the requirements of the chemist rather than those of the optician. The increasing demands of the computer of optical systems for high accuracy in the refractometry of the glasses he has to employ makes it essential that an accurate and at the same time rapid method for the refractometry of glass should be available. The Pulfrich method is the only one as far as the author is aware in which the desired accuracy is approached without the expenditure of an excessively large amount of time on each test. The purpose of these notes is to point out some of the reasons for the defective performance of existing refractometers and to suggest modifications in the design and precautions in the use of the instrument which will help considerably towards the attainment of the theoretical accuracy.

§ 2. POTENTIALITIES OF PULFRICH REFRACTOMETER.

As users of the instrument are aware, settings are made by means of a micrometer tangent screw of which the drum is subdivided to tenths of a minute. Experience shows that when the substance under test is satisfactory as regards homogeneity and surface, settings can be repeated without difficulty to a tenth of a minute. This corresponds approximately to an accuracy of 0·00001 in the refractive index of the specimen; consequently the instrument ought, if free from all sources of error other than the uncertainty of setting, to give results accurate to one unit in the fifth decimal place.

In practice one finds that instead of realising this accuracy results are frequently in doubt in the fourth place, while the dispersions, *i.e.*, the differences between the indices for different wave-lengths, although measured on the run of the micrometer, are not obtainable with a *certainty* of better than three units or thereabouts in the fifth place. This, of course, is apart from any systematic errors due to erroneous constants for the block of the instrument, calibration errors in the scales, faulty disposition of the various parts, &c. In the presence of such errors much more serious discrepancies may arise.

§ 3. NON-SYSTEMATIC ERRORS.

Instrumental Defects.—One of the worst features of the Zeiss Pulfrich refractometer is the coarse and irregular dividing of the scale of the circle. This makes it impossible to obtain satisfactory observations when co-ordinating the circle readings with those of the micrometer, and contributes largely to the uncertainty in the final result for the absolute index. This can only be reduced to within reasonable limits by tedious repetition of circle readings—more than there is time for in routine test work. The dividing of the circle and vernier should be of the finest possible quality, and the diameter of the circle could profitably be increased to 6 inches or even more.

The micrometer screw and fittings are of poor design. The split nut in which the screw works is much too short, with the result that it is impossible to adjust the tightness so as to have the screw moving without shake, but at the same time freely, over its whole range. Moreover, on account of the leverage of the screw, it soon works loose after being tightened up ; so one is either working with a rather tight screw or a shaky one—both of which conditions are inimical to accuracy ; although, with proper caution to avoid backlash, the shaky condition is the lesser evil. These are the chief sources of non-systematic error due to instrumental defects. They are serious in the existing model of the refractometer, but could be got rid of almost entirely by proper design and workmanship.

Sources of Error in Using the Instrument.—The important points to receive attention in making a test are the fitting of the glass specimen on the block of the instrument, and the proper adjustment of the source of light. With regard to the former, it is clearly essential in order that refraction may be regarded as taking place at a single surface separating the specimen from the block, that the film of liquid used to give optical contact should have parallel sides.

This, of course, is generally realised, and the specimen is usually adjusted until the interface appears all of one interference colour when viewed by white light from a lamp suitably placed, or until only a few fringes *running parallel to the plane of refraction* are seen. Slight want of parallelism in a direction transverse to the plane of refraction will have a negligible effect on the angle of refraction since the deviation is in the neighbourhood of a minimum with respect to this adjustment. Any want of parallelism in a direction parallel to the plane of refraction gives rise to serious errors, except in the case of glasses of index quite near that of the α-mono-bromnaphthaline usually employed for the liquid film. In the case of glasses of low index, such as boro-silicate crowns, hard crowns, &c., the position of the critical edge is very sensitive to errors of parallelism in the film.

It is nevertheless possible to make the adjustment with the necessary accuracy ; but it is important to note that it is by no means certain to remain correct throughout the test of a specimen. A slight settling of the specimen, produced or aggravated by any vibration such as that caused by the interrupter of the induction coil, may be sufficient to cause several transverse fringes to develop at the interface during the measurements. The disposition of the fringes should always be examined after a test to detect any shift that may have occurred ; or alternatively the first line measured may be repeated at the end.

In either case if shift is detected it is necessary to re-adjust the specimen and repeat all observations. The possibility of such a shift may be reduced by using a very small drop of the liquid and squeezing most of it out in working the surfaces together. Unfortunately this accelerates considerably the rate of deterioration of the block due to scratches on the surface, and it is necessary to compromise somewhat in the quantity of liquid employed.

A method of minimising this difficulty is to use a liquid with index only a little greater than that of the glass under test. It can be shown (*see* later, p. 174) that the error which will be made in measuring an index μ_1 with a liquid film of index μ_2

and angle θ seconds is $\frac{\theta\mu_2}{2\cdot06}\sqrt{1-\left(\frac{\mu_1}{\mu_2}\right)^2}$ units in the fifth place. The index of α-monobrom-naphthalene is 1·66, so that in measuring a glass of index 1·52, the error for each second of film angle is $\frac{1\cdot66}{2\cdot06}\sqrt{1-\left(\frac{1\cdot52}{1\cdot66}\right)^2}=0\cdot3$ units, *i.e.*, to throw the result out by 0·00001 a film angle of 3 seconds is required. The interference fringes are usually viewed fairly obliquely; so, allowing for this and for the index of the film, an angle of 3 seconds would correspond to about two or three transverse fringes in an interface of 15 mm.

The effect of film angle will clearly be reduced the more nearly μ_2 equals μ_1. To obtain a serious advantage, however, the approximation must be fairly close, because of the way in which the square root term varies with μ_2. Thus for $\mu_2=1\cdot55$ instead of 1·66 the error for a given angle of film is only halved. To reduce it to a quarter, μ_2 must be reduced to a little under 1·53, which is very close to the index of the glass. On the other hand, if μ_2 is too close to μ_1 there is a greatly increased difficulty in seeing the fringes owing to the small amount of reflection at the μ_1—μ_2 surface.

Liquids that prove satisfactory in use with the usual types of glass are :—

(*a*) For Borosilicate Crowns and Hard Crowns :—oil of cloves, oil of sassafras.

(*b*) For Baryta Light Flints :—nitro-benzol, oil of aniseed.

(*c*) For Medium Barium Crowns :—α-monobrom-naphthalene diluted with xylol (proportions may be readily found by trial).

(*d*) For Heavy Flints :—α-monobrom-naphthalene.

A further advantage of using a liquid with an index near that of the glass under test is that it minimises somewhat the defects of focus and definition due to imperfect planeness of the surface of the specimen.

Another matter requiring reasonable care is the adjustment of the source of light. It is sometimes possible to get a spurious sharp edge in the field of view which is sensibly displaced from the proper critical edge. At other times difficulty is experienced in getting a sharp edge at all. The only sure guide to the proper adjustment of the vacuum tube and condensing lens is experience.

Another source of uncertain readings is chromatic parallax. This chiefly affects the blue end of the spectrum. Its cause and cure are discussed in a previous Paper by the author* and need not be gone into in detail here. It was there stated that part of this effect is due to light scattered by the matt surface surrounding the polished horizontal face. This could be greatly reduced by polishing the curved surface as well as the plane. Another suggestion for the elimination of chromatic parallax in this instrument involves the use of a different type of graticule from that usually fitted. The new graticule consists of two vertical lines separated by such a distance that they subtend about seven-tenths of a minute at the centre of the object-glass. These lines are parallel to the edges of the coloured bands at the centre of the field. The setting is made by bringing the bifilar completely on to the coloured band and adjusting until the space between the edge and the wire most remote from it is bisected by the other wire. The wires being illuminated directly by the light of the band, chromatic parallax will not make itself evident unless the proportion of scattered light in the field is very strong. With this setting we are further freed from the difficulty, sometimes strongly felt with the ordinary cross-lines, that if the field is free from scattered light the part of the lines which are on the dark side of the edge are not always seen sufficiently well for proper settings to be made. There is one unfortunate drawback associated with this type of graticule, and that is that it cannot always be employed with the sodium line. There is a point to be observed in making measurements with sodium light on the Pulfrich refractometer that is not generally realised. In the first place one would hesitate to mention the obvious fact that the critical edge in the case of sodium light corresponds to the D_1 line,† but for the prevalent habit of assuming that it is the mean index for D_1 and D_2 that is given. Thus the tables issued by Zeiss and Hilger are stated to apply to the mean of the sodium lines instead of to D_1, as they ought to be.

For some specimens the instrument does not resolve the D lines, but in many cases they are resolved and can be seen as separate bands if the width of the bands is cut down sufficiently by means of a suitable shutter. When comparatively broad overlapping bands are employed, as in the ordinary use of the instrument, only a single band is seen, which is of double brightness except for the region near the edge where the light is due to D_1 alone. One does not, however, see the two edges ; the eye simply averages them up as a single band with a soft instead of a sharply defined edge. When crosslines are set on an edge of this character the eye appears to accept as the edge the region where the intensity is falling off most rapidly, and the setting is made on a point inside the true edge of the outer component by an amount which seems to vary considerably with different individuals. Thus if one carefully sets the crosslines on the edge of the sodium band when it is broad (in the case of a specimen which can resolve the lines) and then gradually cuts down the width of the band, the setting will remain apparently correct until the bands are nearly resolved. The edge will then appear to move relative to the crosslines until, when the two components are completely resolved, it will be seen that the cross-lines are quite away from D_1. In many cases they will be much nearer D_2. As mentioned already, the setting is liable to relatively large variation of personal error, and so cannot be allowed for in the form of a correction. The only way to make satisfactory measurements with sodium is to cut down the width of the band until the components are resolved if it is possible to resolve them.* This, of course, prevents the use with this line of the bifilar graticule, which requires an appreciable width of band. To overcome this difficulty a compound graticule consisting of the ordinary diagonal cross-lines in addition to the bifilar was devised, Fig. 1. The two sets of lines are fairly close, so that neither is far from the centre of the field of view. The actual dimensions suggested for a focal length of $4\frac{1}{2}$ inches as the result of experiments with different graticules were: separation of components of bifilar=0·025 mm.; distance AB=0·25 mm.*

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* Proc. Phys. Soc., Vol. XXIV., p. 330, *et seq.*

† Except in the rare cases when the dispersion of the substance under test is greater than that of the block in which case it is the D_2 line that gives the critical edge.

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* This will not be found to be at all a convenient process if the sodium light is obtained from a flame. In the first place the bands from a flame when cut down fine are very faint, and in the second place owing to the absence of background the cross-wires cannot be seen. For this reason the author prefers to obtain the sodium light from the vacuum tube used for the hydrogen lines, running it for the purpose at a pressure of about 3–5 cm. of Hg. This gives an intense sodium band when it has been running for a short time with a heavy discharge. The width of the band can then be controlled by means of the shutter in front of the condensing lens, while the necessary background for rendering the crosswires visible is supplied by the secondary hydrogen spectrum. Since the observations on the hydrogen lines themselves require exhaustion to about 1 mm. or less, somewhat special apparatus is necessary. The apparatus now used at the Laboratory was shown at the Optical Society's Exhibition on Jan. 11, 1917, and is described in the "Transactions" of the Optical Society, Vol. XVIII., 1917, pp. 22S and 23S.

In use, the bifilar setting would be employed for the G' line, for which it is primarily intended, and probably also for F and C; although for these lines it would not likely possess

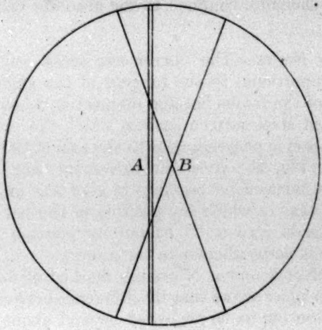

FIG. 1.—COMPOUND GRATICULE (*not* to scale).

any marked advantages over the other. The ordinary cross would be employed for the sodium lines and for the zero reading, the constant difference between the two settings being determined once and for all on some line such as F, which is suited to both.

§ 4. SYSTEMATIC ERRORS.

Centering Error.—There is no ready means of testing the circle of the ordinary Pulfrich refractometer for centering error. If this is present, as it is almost certain to be to a greater or less extent, incorrect angles of emergence will be obtained. A second vernier diametrically opposite to the first should be fitted, so that by taking the mean of their readings centering error may be eliminated.

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* This graticule is described in the present paper with the distinct reservation that it has not yet been possible to experiment with it under the actual conditions of illumination, definition, etc,, of the refractometer. The experiments from which these dimensions were deduced were made on an ordinary spectrometer, the type of setting being imitated as closely as possible. The refractometer itself is usually so fully employed that the fitting of experimental graticules has not so far been feasible.

Unfortunately the methods by which centering errors can be measured when only one vernier is provided are not such as are usually available except in specially equipped laboratories. Of course, imperfect centering of the circle will not affect the dispersions measured on the micrometer screw. This error is therefore confined to the absolute values of the indices.

Micrometer Errors.—The micrometer screw readings give quantities proportional to the tangent of the angle between any position of the radius bar and its position when the plane of the polished steel surface against which the point of the micrometer bears is perpendicular to the axis of the latter.

The curve (Fig. 2*a*) gives the correction that has to be applied to the micrometer readings to give true angles. The case taken is that in which the position of the bar when the micrometer reads zero is 2° behind its position when the bearing plane is perpendicular to the screw.

The Pulfrich instrument is usually marked off for a range of 5° ; but the figure shows that the difference between tangent and angle is too big to be neglected beyond about 3°. It is convenient for glass testing to mark off an extra 2°, so as to increase the range to 7°. The correction becomes rapidly large at these angles.

There is a further source of error which may arise in the micrometer readings, and which may easily be more serious than those due to taking the tangents for the angles. The point involved, though well known to the makers of astronomical instruments for example, does not appear in all cases to receive the attention that it should where tangent screws are utilised as micrometers ; and it may be worth while on that account to indicate the nature and magnitude of the effect.

Suppose we have a radius bar CD (Fig. 3*a*) rotating about a centre C and actuated by a micrometer screw of which the axis is in the direction AB. Let $P'Q'$ be the plane of the bearing surface and Cb' the plane through C parallel to it. Let the position of the bar when the micrometer reads zero be as shown, in which it makes an angle θ_0 with the position in which the bearing plane is perpendicular to the screw.

Suppose the screw is now advanced until the perpendicular position is reached. The micrometer registers the length aa', corresponding to an angle (taking tangents for angles) of $\frac{aa'}{Cb}$.

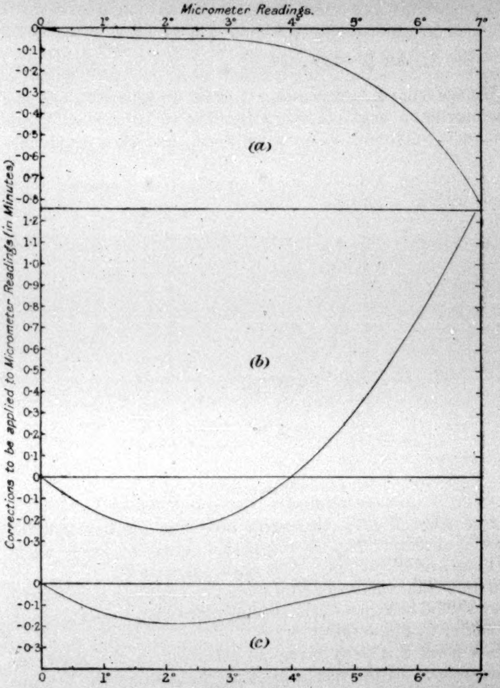

FIG. 2.

The actual angle through which the bar has rotated is $\frac{bb'}{Cb}$. The micrometer reading, therefore, exceeds the true angle by

$$(a'a-b'b)/Cb=(a'b'-ab)/Cb=\frac{ab}{Cb}(\sec\theta_0-1).$$

Thus a correction of $-\frac{p}{r}(\sec\theta_0-1)$ has to be applied to the micrometer reading to give the true angle, p being put for the perpendicular distance from the centre to the bearing plane and r for the perpendicular from the centre to the micrometer axis. It is easy to see that in general for any angle θ measured from the zero of the micrometer, the correction to be applied is

$$-\frac{p}{r}\{\sec\theta_0-\sec(\theta-\theta_0)\}.$$

We see from this expression that as we proceed along the micrometer, a negative correction has to be applied which reaches a maximum value when $\theta=\theta_0$, and then diminishes to zero when $\theta=2\theta_0$, thereafter becoming positive and increasing rapidly. Fig. 2*b* shows this correction curve when $p=1$ cm. and $r=8{\cdot}5$ cm.; θ_0 is again taken as 2°.

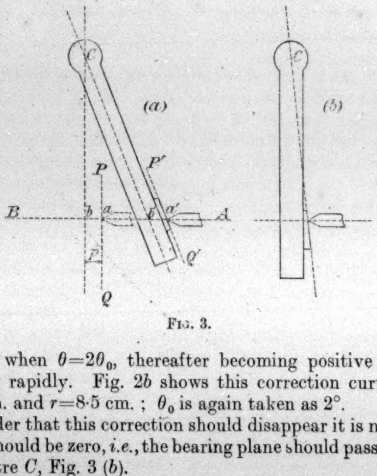

Fig. 3.

In order that this correction should disappear it is necessary that p should be zero, *i.e.*, the bearing plane should pass through the centre C, Fig. 3 (*b*).

This point has been given no attention in the Zeiss refractometer. In the specimen at the Laboratory the bearing plane instead of sloping so as to pass through the centre as in Fig. 3 (*b*) actually slopes away from it, the result being a large value of p. Fig. 4 shows the *total* correction curve which has to be applied to the micrometer readings of this instrument. Since this includes the tangent corrections corresponding to Fig. 2(*a*) as well as the error under discussion, to get the curve for this error alone we have to raise curve 4 by the ordinates of a curve similar to Fig. 2(*a*). When we see that this error, which ought to be zero throughout, reaches nearly 2 minutes at 7°.

Unfortunately it is not easy to determine the curve of correction to a micrometer screw in the absence of special apparatus. One method by which it can be carried out on the instrument itself is to clamp the radius bar to the circle in such a position that when the micrometer reads zero there is an approximate coincidence between the zero of the vernier and a scale division on the circle. The coincidence is then made accurately by means of the tangent screw, and the micrometer reading noted. The screw is then advanced until the next scale division is opposite the vernier zero and so on. In this way a series of micrometer readings corresponding to every half degree is obtained. The measurements must be repeated at several different parts of the circle to average out irregularities in the circle graduations. If these graduations were of satisfactory fineness this would be a comparatively good method of calibrating the micrometer, provided, of course, that the patience in repetition which is essential to all standardisation methods is applied. With the Zeiss instruments, however, owing to the poor quality of the dividing the error of a single observation is so great that the amount of repetition

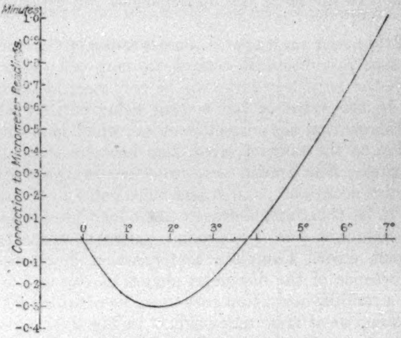

FIG. 4.—PULFRICH REFRACTOMETER.
Total correction curve to an actual micrometer.

required to get satisfactory results is excessive, and one would not use the method except where others are not available.

But although this method would not be employed in a standardising institution, it certainly gives the user of the instrument a means, even if it is a tedious one, of determining this important correction which is absolutely vital to dispersion measurements. It will sometimes be found that in calibrating the screw it is impossible to get successive calibrations to agree satisfactorily. Irregularities of this kind are often traceable to the vernier rubbing on the edge of the circle. If the vernier is set back sufficiently to allow perfectly free motion of the circle the discrepancies will probably disappear.

Figs. 2 (*b*) and 4 show that if there is excess of the error due to the cause just discussed—which we may call the p error for short—the correction curve will be very steep at the higher angles. In the event of the p error being entirely absent, Fig. 2 (*a*) shows that the correction curve, which is in this case simply due to the tangent error, also becomes steep at the higher angles. Now a steep correction curve is always difficult to determine accurately, and is also more liable to be misread in use. It is, therefore, better to introduce intentionally a suitable amount of p error to produce partial compensation of the tangent error. Complete compensation is clearly impossible because of the dissimilar shapes of curves 2 (*a*) and 2 (*b*), but a residual correction curve may be obtained which is nowhere steep or of large amplitude. In Fig. 2 (*c*) is plotted the resultant correction curve obtained by introducing sufficient p error to neutralise the tangent error at 6° from the zero of the micrometer. The value of p/r required when $\theta_0=2°$ is 0·067. This gives a much more satisfactory correction curve over the 7° range than Fig. 2 (*a*) alone. A flat curve like this could then be compensated completely by suitably shaping the edge of the plate against which the drum is read as in Fig. 5.

Before leaving this question it may just be mentioned that if p is of opposite sign to the cases considered here, *i.e.*, if the plane of the bearing surface passes the centre on the side remote from the micrometer the resulting errors will be of opposite sign to those of Fig. 2 (*b*). This, however, is not likely to be so frequently met with as the other case.

Filling the Aperture.—The next point to be considered in connection with possible systematic error is the manner in which the beam of light enters the telescope. The refracted light all comes from the circular horizontal face of the block. Owing to the obliquity produced by the two refractions, the section of the beam when it reaches the telescope is a somewhat narrow ellipse of which the minor axis is parallel to the plane of refraction, and is the effective aperture of the beam in this plane. The effective aperture is zero for angles of emergence from the second face of the block of 0 or 90°, and has a maximum value of roughly one-third the diameter of the interface at an intermediate angle. The telescope is fitted with an elliptical stop in front of the object-glass, which is intended to be just about the size of the maximum aperture of the beam, but is usually a bit larger. Now the pencil of light which enters the telescope can only be symmetrical with the optic axis if the central ray of the beam passes through the centre of the stop or else if the width of the beam is sufficiently in excess of the width of the stop to ensure that the latter is completely filled.

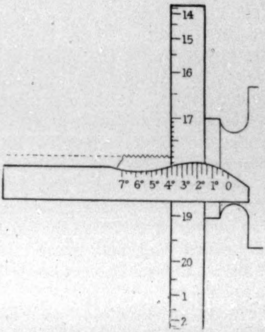

FIG. 5. (About twice full size.)

From the nature of the case it is impossible for the central ray of the emergent beam to pass centrally through the stop for all angles of emergence, because these central rays, produced backwards, do not radiate from any single point.

There is thus no point at which the centre of rotation can be situated, such that the axis of the telescope can be brought into *coincidence* with the central ray of the emergent beam for all angles of emergence.

The only method of ensuring that a symmetrical portion of the objective is used is to increase the diameter of the circular face of the block so that the width of the emergent beam is sufficiently in excess of the aperture of the telescope to fill it at all angles within the practical range.

In the ordinary refractometer this is never the case. When the telescope is set on a spectrum line, if the exit pupil be examined with a magnifier the elliptical section of the light beam and its relation to the stop can be easily seen. The illuminated ellipse will usually be smaller than the stop and will probably lie towards one side of it.

When the central ray of the beam which enters the telescope does not pass through the centre of the objective, errors will arise in the measured position of the image unless the focal plane coincides exactly with the plane of the cross-lines. There is a variety of causes from which slightly incorrect focus may arise in practice. In the first place the adjustment may not be quite correct for any wave-length, while it certainly can not be correct for all wave-lengths with the ordinary doublet lens employed. Also slight defects of planeness of the surface of the specimen under test appreciably affects the focus. Thus to be quite safe from error the aperture must be fully or at least symmetrically filled with light. If such errors are present they may enter into the dispersion measurements, since the focal defects are not necessarily the same for all wave-lengths.

Fortunately in existing refractometers it is usually possible to make the beam symmetrical with the stop by adjusting the height of the block on the pillar while examining the exit pupil. Every time this is done the condenser and source of light must be adjusted to suit.

Adjustment of Telescope and Block.—It is essential that the axis of the telescope should be perpendicular to the axis of rotation, and also that the refracting edge of the block should be parallel to the axis of rotation. The former adjustment can be very easily verified by removing the block and resting on the top of the pillar a strip of brass to which a piece of plane glass has been fixed, approximately at right angles, by means of wax or plasticene. The telescope is then brought up until the autocollimated image from the front face is in the field, and the brass plate is then adjusted by hand in the horizontal plane until exact coincidence can be obtained. The telescope is then rotated through 180°, so as to obtain the image reflected from the back of the glass. If the adjustment of the axis is correct exact coincidence can be obtained without altering the position of the plate. If, however, the image is nearer to or further from the centre of the field than it should be, half the defect must be remedied by altering the adjustment of the glass and half by the adjustment of the telescope. Unfortunately there is no proper provision for this adjustment, so that if any is really necessary it must be done by packing under the flange with thin tin foil.

When the telescope is correctly fitted the block should give normal reflection from each of the faces. The adjustment of the vertical face is usually left just a little bit out to give a better type of setting for the zero readings. This degree of incorrect adjustment is, of course, negligible, but the adjustment of the horizontal face is frequently found to require improvement. There is no adjustment provided for this, and here again packing must be resorted to if necessary.

Optical and Geometrical Properties of Block.—The final direction of the emergent beam depends on the refractive index of the block and also on its angle; and it is not generally found that either of these constants is sufficiently near the values adopted in computing the tables for use with the instrument. It is necessary to check both the index of the block and the angle, and to apply corrections if these differ from their nominal values.

The determination of the index of the block can only be performed by determining the angles of emergence when a specimen of known properties is placed on it. It is obvious at the outset that to accomplish this calibration efficiently is a matter of some difficulty. The requirements as to a standard substance are very severe. If a glass specimen is employed it must be one of which the indices have been determined by an absolute method to a certain accuracy of 1 in the fifth decimal place. This requires a spectrometer outfit of the highest precision, and there are very few laboratories in which equipment nearly good enough for the purpose is available. The alternative is to use some substance such as quartz, for which accurate data have been obtained by various experimenters. Unfortunately, when the available data for quartz is consulted very considerable differences between the results of different observers are found. Some do not specify whether right or left handed quartz is referred to, although there appears to be an appreciable difference between their refractive properties. It is somewhat difficult, therefore, to decide what weight to give to different determinations. The values provisionally adopted at the laboratory for the ordinary ray are :—

$$n_C=1{\cdot}54187,\quad n_{D_1}=1{\cdot}54421,\quad n_F=1{\cdot}54966,\quad n_{G'}=1{\cdot}55394,$$

based on the results of Martens, Gifford, Müller, van der Willigen, Macé de Lépinay, Dufet, Sarasin, Quincke and Mascart.

These are only accepted, however, pending the completion of experiments that are in progress. At present the author inclines to the belief that variations of several units in the fifth place may be found in the indices of specimens of quartz obtained from different localities. This might explain the difference between Col. Gifford's figures and those of most other observers.

Much closer agreement is to be found among the various determinations for distilled water ; but the high temperature coefficient of liquids and also the fact that water cannot be used with the denser blocks renders this substance unsuitable for standardisation purposes.

We are thus faced at the outset with considerable difficulty in obtaining a standard substance with sufficiently well-established constants to give the requisite accuracy in the constants of the block.

In order to overcome this difficulty the author has suggested the use of blocks with smaller angles than the customary 90°. It is clear that if the angle of the block is less than $2\sin^{-1}\frac{1}{\mu}$, where μ is its refractive index, there will be an emergent beam corresponding to light entering the top surface at grazing incidence in air, and the index of the block can be measured directly on the instrument without reference to any intermediate standard substance.

§ 5. THE GENERAL CRITICAL ANGLE METHOD.

In order to understand fully the advantages or otherwise associated with this departure it is desirable to examine in detail the problem of refraction by a system such as we have in the Pulfrich refractometer.

Suppose we have a prism of index μ_2 and angle θ separating two media of μ_1 and μ_3. Let *AB*, Fig. 6, be a ray of light incident at grazing incidence on the first face of the prism, *BCD* being its subsequent path. The author finds it convenient in refraction problems to adopt the usual convention that all angles are positive when measured counter-clockwise from the appropriate reference lines. For rays of light the reference lines are the normals at the points of incidence or emergence as the case may be, while for prism angles, the normal to the surface first encountered is taken as the reference line with respect to which the direction of the other normal is defined. In Fig. 6 all the angles are positive on this convention. Taking the signs of the angles into account in this way has the disadvantage that some of the usual formulæ of refraction by prisms have the signs of certain quantities reversed; but it has the much greater advantage of preventing confusion in many practical cases in which doubt as to sign may arise if one considers only the magnitude of the angles as is usually done in evolving prism formulæ.

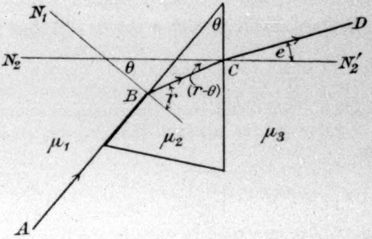

FIG. 6.

Applying the laws of refraction to the two surfaces, we have

$$\frac{\sin(r-\theta)}{\sin e}=\frac{\mu_3}{\mu_2} \text{ and } \frac{\mu_1}{\mu_2}=\sin r,$$

$$\begin{aligned}&=\sin\{\theta+(r-\theta)\},\\ &=\sin\theta\cos(r-\theta)+\cos\theta\sin(r-\theta),\\ &=\sin\theta\sqrt{1-\left(\frac{\mu_3}{\mu_2}\right)^2\sin^2 e}+\cos\theta\frac{\mu_3}{\mu_2}\sin e,\end{aligned}$$

$$\text{whence}\quad \frac{\mu_1}{\mu_3}=\sin\theta\sqrt{\left(\frac{\mu_2}{\mu_3}\right)^2-\sin^2 e}+\cos\theta\sin e\ . \quad (1)$$

Before applying this to the Pulfrich system as a whole we may consider it as applying to the three media—specimen, liquid film, block; in which case μ_1 is the index of the specimen, μ_2 that of the liquid film, θ the angle of the film, and e the angle of refraction *into* the block.

If the adjustment of the film is correct so that $\theta=0$

$$\sin e=\mu_1/\mu_3,$$

as in the theoretical case with no film present. If, however, the adjustment is imperfect, this will not be the case. Equation 1 becomes for very small values of θ

$$\frac{\mu_1}{\mu_3}=\theta\sqrt{\left(\frac{\mu_2}{\mu_3}\right)^2-\sin^2 e}+\sin e(1-\tfrac{1}{2}\theta^2)$$

$$\sin e=\frac{\mu_1}{\mu_3}-\theta\sqrt{\left(\frac{\mu_2}{\mu_3}\right)^2-\left(\frac{\mu_1}{\mu_3}\right)^2},$$

neglecting terms involving higher powers of θ than the first.

Whence $$\mu_3\sin e=\mu_1-\mu_2\theta\sqrt{1-\left(\frac{\mu_1}{\mu_2}\right)^2}.$$

Now, for a given direction of the emergent beam we would deduce a value μ_1' for μ_1, such that $\mu_1'=\mu_3\sin e$; so that the error

$$\mu_1-\mu_1'=\mu_2\theta\sqrt{1-\left(\frac{\mu_1}{\mu_2}\right)^2}=\mu_2\frac{\theta}{2\cdot06}\sqrt{1-\left(\frac{\mu_1}{\mu_2}\right)^2},$$

where $\mu_1-\mu_1'$ is measured in units in the fifth decimal place and θ is in seconds of arc.

In considering refraction by the whole system we neglect the film and take μ_2 in equation 1 as the index of the block and θ its angle. μ_3 is then the index of air and may be put $=1$. The equation then becomes

$$\mu_1=\sin\theta\sqrt{\mu_2^2-\sin^2 e}+\sin e\cos\theta, \quad . \quad . \quad . \quad (2)$$

which is the relation connecting the index of the specimen (μ_1) with the index and angle of the block and the angle (e) of emergence from the last surface in the general critical angle method.

It should be noted in applying the formula to particular examples that e is positive if the emergent ray is on the same side of the normal as the refracting edge, and is negative if it is on the side of the normal remote from the edge.

In the particular case in which $\theta=90°$, $\mu_1^2=\mu_2^2-\sin^2 e$, which is the relation for the ordinary Pulfrich refractometer.

If we differentiate the right-hand side of equation 2 with respect to e, θ and μ_2 being constant, we find

$$\frac{\partial \mu_1}{\partial e} = \frac{1}{2}\,\frac{\sin\theta}{\sqrt{\mu_2^2-\sin^2 e}}(-2\sin e\cos e)+\cos e\cos\theta$$

$$=\frac{\cos e}{\sqrt{\mu_2^2-\sin^2 e}}\left\{\cos\theta\sqrt{\mu_2^2-\sin^2 e}-\sin e\sin\theta\right\},$$

which, after transformation and substitution from equation 2, reduces to

$$\frac{\cos e}{\sqrt{\mu_2^2-\sin^2 e}}\sqrt{\mu_2^2-\mu_1^2}.$$

The inverse of this quantity, $\frac{\partial e}{\partial \mu_1}$, measures the sensitivity of the measurements. It is convenient to express sensitivities in terms of seconds of arc per 0·00001 change in μ_1. Calling this quantity S_{μ_1},

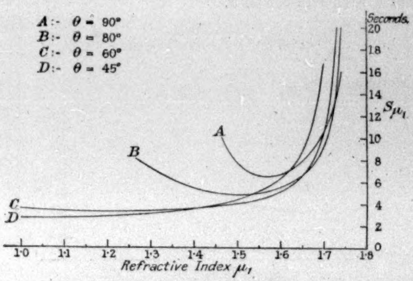

Fig. 7.—Pulfrich Refractometer.
Sensitivity curves for blocks of different angles.

$$S_{\mu_1}=2{\cdot}06\frac{\partial e}{\partial \mu_1}=\frac{2{\cdot}06\sqrt{\mu_2^2-\sin^2 e}}{\cos e\sqrt{\mu_2^2-\mu_1^2}} \quad . \quad . \quad . \quad . \quad (3)$$

For any value of μ_2, S_{μ_1} can be calculated for various values of μ_1 for any prism angle, the corresponding value of e being found from equation 2.*

In Fig. 7 a series of curves are drawn showing for the case

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* Equation 2 is in its most convenient form for deducing μ_1 from known values of e and θ. For the inverse process of deducing the value of e corresponding to a given μ_1 it can be reduced to the form

$$\sin e=\mu_1\cos\theta-\sin\theta\sqrt{\mu_2^2-\mu_1^2}.$$

in which $\mu_2=1{\cdot}75$ (a usual value) how the sensitivity varies with the index under test (μ_1) for blocks of different angles. The sensitivity is in all cases high for indices near that of the block, diminishing to a minimum value which is lower, and occurs at a lower value of μ_1 the smaller the angle.

Thus with a 90° angle the minimum occurs at $\mu_1=1{\cdot}57$ approx. and is about 6 seconds : for $\theta=80°$ the minimum is at $\mu_1=1{\cdot}5$ and is under 5 seconds ; for $\theta=60°$ the minimum is not reached until μ_1 is in the neighbourhood of 1·2 and is just over 3 seconds, while for $\theta=45°$ the minimum is never reached. These results will be discussed later in conjunction with others.

Effect of Error in θ.—If an erroneous value is assumed for the angle of the block, the corresponding error in the calculated value of μ_1 will be proportional to the partial differential of equation 2 with respect to θ.

$$\frac{\partial\mu_1}{\partial\theta}=\cos\theta\sqrt{\mu_2^2-\sin^2 e}-\sin e\sin\theta,$$

which reduces to $\sqrt{\mu_2^2-\mu_1^2}$.

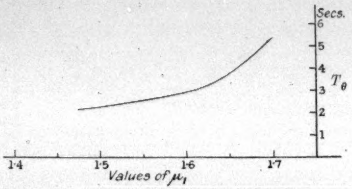

FIG. 8.—PULFRICH REFRACTOMETER.
Error in angle of block which will cause error of 0·00001 in μ_1. $\mu_2=1{\cdot}75$.

If we denote by T_θ (which we may call the tolerance in prism angle) the error in θ, measured in seconds, which will throw out the value of μ_1 by 0·00001, we may write

$$T_\theta=2{\cdot}06\frac{\partial\theta}{\partial\mu_1},$$
$$=\frac{2{\cdot}06}{\sqrt{\mu_2^2-\mu_1^2}}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is always positive, so that if the assumed value of θ is too large, the deduced value of μ_1 will be too large and vice versa. Also we see that T_θ is independent of the angle of the block, *i.e.*, the error in θ which will affect μ_1 by a given amount is independent of the shape of the block.

In Fig. 8 the way in which the tolerance varies with the index under test is shown. We see from it that the effect of an error in the angle of the prism diminishes the higher the index under test.

For the lightest glasses ($\mu_1=1{\cdot}5$) an error of 2 seconds in the angle of the block will throw out the index by one unit. While even if we confine the use of this block ($\mu_2=1{\cdot}75$) to glasses of 1·6 and over, as would usually be done in practice, the value of T_θ is only 3 seconds. It is thus clearly necessary that the measurements of the angle of the block should be made with the utmost accuracy.

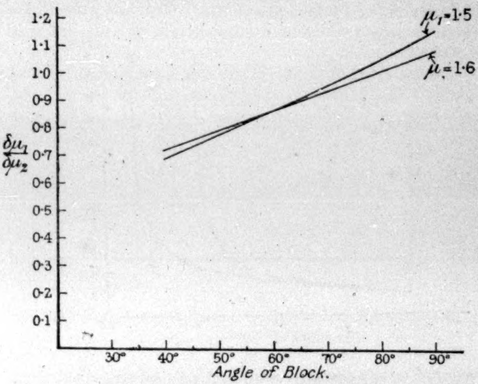

FIG. 9.—PULFRICH REFRACTOMETER.

$\frac{\partial\mu_1}{\partial\mu_2}$ for different angles of block. $\mu_2=1{\cdot}75$.

Effect of Error in μ_2.—If an erroneous value is assumed for μ_2, the index of the block, the error produced will be proportional to $\frac{\partial\mu_1}{\partial\mu_2}$.

$$\frac{\partial\mu_1}{\partial\mu_2}=\frac{\mu_2\sin\theta}{\sqrt{\mu_2{}^2-\sin^2 e}}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

In the case of the ordinary Pulfrich instrument this reduces to μ_2/μ_1.

In Fig. 9 the value of $\partial\mu_1/\partial\mu_2$ is shown for different block angles for two values of μ_1. The effect of an error in μ_2 diminishes as the angle of the block is diminished.

Measurement of μ_2 with Air as Standard.—Before discussing the foregoing results, let us consider the method of calibrating the block without an auxiliary standard substance. As mentioned already, the largest angle with which this would be possible is $2 \sin^{-1}\frac{1}{\mu_2}$, which for $\mu_2=1{\cdot}75$ is about $69\frac{3}{4}°$, while for the lower index 1·6 it is about $77\frac{1}{3}°$.

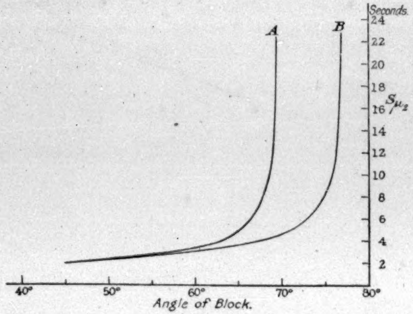

Fig. 10.

S_{μ_2}=number of seconds change in emergent angle for change of 0·00001 in μ_2 while measuring μ_2 with air as standard substance.

A : $\mu_2=1{\cdot}75$. *B* : $\mu_2=1{\cdot}6$.

If $\mu_1=1$, equation 2 may be reduced to

$$\mu_2^2=1+\left(\frac{\cos\theta-\sin e}{\sin\theta}\right)^2, \quad . \quad . \quad . \quad . \quad (6)$$

a form suitable for calculating μ_2 from observed values of e, or

$$\sin e=\cos\theta-\sin\theta\sqrt{\mu_2^2-1}\ , \quad . \quad . \quad . \quad (6a)$$

which is more convenient for calculating the angle of emergence for a given value of the index of the prism.

The sensitivity of this method may be denoted by S_{μ_2} and is obviously, in the same units as before, given by

$$S_{\mu_2}=2{\cdot}06\,\frac{\partial e}{\partial\mu_2}.$$

Differentiating equation 6

$$2\mu_2\frac{\partial\mu_2}{\partial e}=-2\left(\frac{\cos\theta-\sin e}{\sin\theta}\right)\frac{\cos e}{\sin\theta}=-2\sqrt{\mu_2^2-1}\,\frac{\cos e}{\sin\theta},$$

or

$$S_{\mu_2}=-\frac{2{\cdot}06\mu_2\sin\theta}{\cos e\sqrt{\mu_2^2-1}}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

S_{μ_2} is clearly always $-$ve, *i.e.*, the ray is always deviated further away from the refracting edge for an increase of μ_2. In Fig. 10 S_{μ_2} is shown for different angles in the cases of $\mu_2=1{\cdot}75$ and $\mu_2=1{\cdot}60$.

It is infinite for the maximum possible angle, decreasing with great rapidity as the angle is diminished by a few degrees, and then varying much less rapidly for still lower angles.

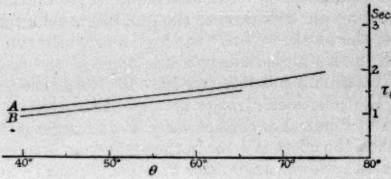

Fig. 11.

τ_θ= Error in angle of block which will produce error of 0·00001 in μ_2.

$A: \mu_2=1{\cdot}6$. $B: \mu_2=1{\cdot}75$.

Effect of Angle Error in Measuring μ_2.—From equation 6 we obtain

$$2\mu_2\frac{\partial\mu_2}{\partial\theta}=2\left(\frac{\cos\theta-\sin e}{\sin\theta}\right)\left[-\frac{\sin\theta}{\sin\theta}-\frac{(\cos\theta-\sin e)}{\sin^2\theta}\cos\theta\right]$$

$$=-2\sqrt{\mu_2^2-1}\left(1+\frac{\cos\theta}{\sin\theta}\sqrt{\mu_2^2-1}\right),$$

whence

$$\tau_\theta=2{\cdot}06\frac{\partial\theta}{\partial\mu_2}=\frac{-2{\cdot}06\mu_2}{\sqrt{\mu_2^2-1}+(\mu_2^2-1)\cot\theta}, \quad . \quad (8)$$

where τ_θ is the tolerance in angle for measurement of μ_2.

In Fig. 11 the variation of this function with θ for indices of 1·75 and 1·60 is shown. We see that the accuracy demanded in the measurement of the angle increases as the angle is diminished.

§ 6. Discussion of Results.

We are now in a position to study the effect of varying the angle of the block on the general utility of the instrument. Referring in the first place to the sensitivity curves of Fig. 7, we see that owing to the way the curves intersect one another a block of any particular angle is less sensitive than one of a larger angle below a certain value of μ_1 and more sensitive above that value. Thus a 60° block is more sensitive than one of 80° if $\mu_1 > 1{\cdot}65$, and better than 90° if $\mu_1 > 1{\cdot}71$; while a 45° block is better than 60° above 1·425, *i.e.*, for all glasses, better than 80° above 1·54 and better than 90° above 1·62. The effect of errors in μ_2 (*see* Fig. 8) also diminishes the smaller the angle; while the effect of errors in the angle of the block was seen to be independent of the angle. These considerations point to the desirability of using as small an angle as possible.

If we now turn our attention to the problem of determining the index of the block, we find that high sensitivity can only be obtained in this measurement if the angle of the block is nearly the maximum possible (*see* Fig. 10). For the major portion of the practicable range of angle the sensitivity is fairly low and diminishes continuously as the angle is made less. Further, the effect of error in the angle on this measurement increases at lower angles (*see* Fig. 11). Thus from the point of view of accuracy in the standardisation of the block it is desirable to use as large an angle as possible. Moreover, the gain in sensitivity in standardisation which would be effected by choosing a value of θ well on the steep parts of the curves (Fig. 10) much more than outweighs the gain in sensitiveness given by a smaller angle in the measurements of μ_1, subsequently made with the block. We would therefore be led to employ for a given block as large an angle as possible consistent with the existence of an emergent beam when $\mu_1=1$.

There are, however, other considerations to be taken into account. If we use a value of θ just a little under $2 \sin^{-1}\dfrac{1}{\mu_2}$, as suggested by these considerations alone, the aperture of the emergent beam when performing the standardisation measurements would be very small and the definition of the telescope would be very seriously impaired. Also the direction of the beam would pass so far from the centre of rotation of the telescope that it would be impossible to arrange the various parts of the instrument to be suitable for its ordinary use and for the standardisation measurements as well.

There is also the question of the measurement of the angle of the block. While it may be possible with a very accurate spectrometer to measure any angle to the desired accuracy—about 2 seconds—it is certainly a matter of the greatest difficulty in the general case. There are, however, comparatively simple methods whereby this accuracy can be obtained in the case of angles of 90°, 60° and 45°. This important limitation, therefore, reduces the choice of angles, other than 90°, to 60° and 45°. This being so, the balance of advantages undoubtedly lies with a 60° block, and we may now consider just what the requirements are in this case and the likelihood of their being satisfactorily realised. From Fig. 10, in the case of a 60° block, the sensitivity when measuring the index of the block is about 3 seconds per 0·00001. The probable error of a single observation is 0·1 minute, which therefore corresponds to 0·00002. In standardisation measurements a large series of observations ought always to be made, and no difficulty should be experienced in obtaining values correct to less than half the uncertainty of a single observation. The necessary accuracy in μ_2 is therefore quite possible provided, of course, that the mechanical accuracy of the instrument is sufficient.

With regard to error in the angle of the block, we find from Fig. 11 that an error of $1\frac{1}{2}$ seconds will be sufficient to produce an error of 0·00001 in the value of μ_2; while in using the block for the measurement of other substances an error of 2 seconds (Fig. 8) will throw out the value of μ_1 by 0·00001 in the worst case ($\mu_1 = 1{\cdot}5$), provided the correct value of μ_2 is known. But if μ_2 is measured on the instrument itself the effect of a given error in angle on the values of μ_1 subsequently tested on the block will be reduced; for the values of $\frac{\partial \mu_1}{\partial \theta}$ and $\frac{\partial \mu_1}{\partial \mu_2}$ are always positive while the expression for $\frac{\partial \mu_2}{\partial \theta}$ is essentially negative. Thus if we assume too large a value for θ the value of μ_1 will also tend to be too large. But we shall have obtained a value for μ_2 in the standardisation measurement that is too low. This will tend to give a small value for μ_1, and will partially counteract the other effect. Since the errors are small we may treat them independently and add them. If $\delta\theta$ is the error in θ, the error in μ_2 on this account is $\left(\frac{\partial\mu_2}{\partial\theta}\right)\delta\theta$.* The error which this introduces in the subsequent measurement of an index μ_1 with the instrument is

$$\left(\frac{\partial\mu_1}{\partial\mu_2}\right)_2\delta\mu_2=\left(\frac{\partial\mu_1}{\partial\mu_2}\right)_2\left(\frac{\partial\mu_2}{\partial\theta}\right)_6\delta\theta,$$

and is negative if $\delta\theta$ is positive. The error in the opposite direction is $\left(\frac{\partial\mu_1}{\partial\theta}\right)_2\delta\theta$, the total error therefore being the algebraic sum, $\delta\mu_1=\delta\theta\left\{\left(\frac{\partial\mu_1}{\partial\theta}\right)_2+\left(\frac{\partial\mu_1}{\partial\mu_2}\right)_2\left(\frac{\partial\mu_2}{\partial\theta}\right)_6\right\}$,

or to express it in the form hitherto adopted,

$$\frac{\delta\theta}{\delta\mu_1}=\frac{1}{\left\{\left(\frac{\partial\mu_1}{\partial\theta}\right)_2+\left(\frac{\partial\mu_1}{\partial\mu_2}\right)_2\left(\frac{\partial\mu_2}{\partial\theta}\right)_6\right\}}\quad.\;.\;.\;.\;.\quad(9)$$

When equation 9 is evaluated we find that the error in angle required to give a value of μ_1 which is too low by 0·00001 is 5½ seconds for $\mu_1=1{\cdot}5$, 3½ seconds for 1·6 and just under 3 seconds for 1·65. There should be no difficulty in determining the angle of the block to within this limit.

Thus the determination of the constants of the block by measurements independent of those made by the makers appears to be quite practicable if the block is cut to 60° instead of 90° as at present. The advantages of this do not require to be emphasised. Quite apart from the fact that individual blocks vary considerably in their properties from the representative samples tested by the makers, it is clearly undesirable that the accuracy of all subsequent work with the instrument should depend on the accuracy with which the initial determinations have been made, or that it should be impossible to check the maker's values except by reference to an intermediate standard, in view of the difficulties in connection with such standards mentioned earlier in the Paper.

A further important advantage of the 60° block is that the actual angles of emergence are in general smaller than in the case of 90° since the rays emerge more nearly normal to the second face. This greatly diminishes the possible effects of centering error or other circle deficiencies. In fact, over a limited range of index the emergent angles will be sufficiently small that both the dispersion measurements and the zero reading can be obtained on the micrometer, thus obviating in these special cases any reference to the circle at all. Specially simplified, but still accurate, refractometers might be constructed on these lines for use over a small range of index. The properties of the block being selected to give zero emergent angle for an index at the middle of the range, the measurements could all be made on a micrometer working over, say, 10°, no circle or clamping screws being required. Such an instrument might be very useful in certain industrial processes.

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* The suffix is the number of the expression from which the partial differential coefficient is obtained.

Reverting to the complete instrument the loss of sensitivity in the 60° case as compared with the 90° is slight over the range of indices required for glass testing, especially if a block of ordinary flint glass is used for the lightest glasses, as is the usual custom. But even using the denser block (1·75) with the lightest glasses, the sensitivity is still sufficient to give indices to the nearest unit in the fifth place if the settings are accurate to the nearest tenth of a minute.

In the calibration of the ordinary 90° block, by means of an auxiliary substance, the compensation of the effect of an error in θ on the measurements subsequently made with the instrument by the error, which is introduced in the value of μ_2 obtained in the standardisation, also holds. In fact, in this respect the auxiliary substance method of calibration is much superior to the other, because the effect of angle error is zero when measuring an index equal to that of the substance used for calibrating the block.

The equation which corresponds to equation 9 when the value of μ_2 is obtained by measurements of an auxiliary substance other than air is clearly

$$\frac{\delta\theta}{\delta\mu_1}=\frac{1}{\left\{\left(\frac{\partial\mu_1}{\partial\theta}\right)_2+\left(\frac{\partial\mu_1}{\partial\mu_2}\right)_2\left(\frac{\partial\mu_2}{\partial\theta}\right)_2\right\}} \quad . \quad . \quad . \quad . \quad (10)$$

In the case of a 90° block $\left(\frac{\partial\mu_1}{\partial\theta}\right)_2$ reduces to $-\sin e$ and $\left(\frac{\partial\mu_2}{\partial\theta}\right)_2$ is easily shown to be $\frac{\mu_1}{\mu_2}\sin e$, while $\left(\frac{\partial\mu_1}{\partial\mu_2}\right)_2=\frac{\mu_2}{\mu_1}$.

Equation 10 then becomes, in seconds per 0·00001,

$$\frac{\delta\theta}{\delta\mu_1}=\frac{2{\cdot}06.}{\left\{-\sin e+\frac{\mu_2}{\mu_1}\,.\,\frac{\mu_1'}{\mu_2}\,.\,\sin e'\right\}} \quad . \quad . \quad . \quad . \quad (11)$$

where μ_1 and e are the index and emergent angle of the substance under test, while μ_1' and e' are the similar quantities for the standard substance employed in calibrating the block.

We find from equation 11 that in the case of a 90° block of $\mu_2=1{\cdot}75$ which has had its index determined from measurements on quartz, for which $\mu_1'=1{\cdot}54$, that the error in θ required to give an ultimate error of 0·00001 in the measured indices is 44″ for $\mu_1=1{\cdot}5$; ∞ for $\mu_1=1{\cdot}54$; 23″ for $\mu_1=1{\cdot}60$; 11″ for $\mu_1=1{\cdot}65$; and 6″ for $\mu_1=1{\cdot}7$.

Note that when the block is calibrated in situ in this way the tolerance in angle is in the worst case nearly three times what it is in the worst case when μ_2 is obtained by measurements independent of the block. The value of μ_2 should always, therefore, be found in this way, rather than by independent measurements on the glass employed before it is cut, in order to minimise the effect of error in angle on the subsequent measurements. Moreover, when a block has been re-surfaced so that the angle has to be measured anew, the calibration of μ_2 should also be repeated so as to introduce the error in μ_2 corresponding to any error in the θ measurement in order to gain the compensation just discussed.*

In the foregoing treatment we have considered the accuracy in the value of θ, which is required to give an accuracy of one unit in the fifth decimal place in the indices measured on the instrument. It was recently suggested by various experts that the present requirements from the point of view of the optical designer are certainty to 1 unit in the fourth decimal place for the absolute indices, and certainty in the fifth place for the dispersion data.

This standard will clearly be attained with much more latitude in the accuracy of the measurement of θ. As far as the effect on the absolute index is concerned we can multiply the tolerances already obtained by 10. In the case of the 60° block, with its own indices determined on the instrument, we have just seen that the tolerance is in the neighbourhood of 3 seconds for 0·00001 error in μ_1. Thus 30 seconds error would be required to throw μ_1 out by 1 in the fourth place.

It is necessary, however, to see whether an error in θ of this magnitude would affect the dispersion measurements in the fifth place. Suppose we have a block of which the indices for the A' and G' lines—the extreme range of wave-length with which we need concern ourselves—are 1·734 and 1·787 respectively, and that we are measuring on it a crown glass of low indices say 1·496 and 1·510 for the same wave-lengths. These are characteristic values. The error in μ_1 due to 1 second error in θ is $1/T_\theta$ and equals $\sqrt{\mu_2^2-\mu_1^2}/2{\cdot}06$.

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* Another important compensation effected in a similar way by calibration in situ—*viz.*, that of constant personal errors of setting—is referred to in the Discussion.

Substituting the values for the two lines

$$(\delta\mu_1)_{A'}=0{\cdot}425\,;\quad(\delta\mu_1)_{G'}=0{\cdot}462.$$

$$\therefore\ \delta\{(\mu_1)_{G'}-(\mu_1)_{A'}\}=0{\cdot}037 \text{ unit in fifth place.}$$

Thus to produce an error of 1 unit in the fifth place in the dispersion from A' to G' would require an error in angle of $1/0{\cdot}037=27$ seconds.

In a similar way, if we are measuring a hard flint of indices, say 1·610 for A' and 1·643 for G', we find that the tolerance in angle from the point of view of the dispersion is 33 seconds.

In deducing the effect on the dispersions we have neglected the compensating effect of the corresponding errors in μ_2 which are produced if the block is calibrated on the instrument. The values of $\frac{\partial\mu_1}{\partial\mu_2}$ and $\frac{\partial\mu_2}{\partial\theta}$, from which the compensation arises, vary so very little over the range of indices involved in the dispersion that the compensating term is the same for both wave-lengths. Thus, while the actual error in index due to a given error in θ is profoundly modified by measuring μ_2 on the block itself, the effect on the dispersion is unaffected.

We may thus say that for a 60° block if the error in the angle of the block is less than, say, 25 seconds the errors from this cause will not exceed 0·0001 in absolute index or 0·00001 in dispersion. The same figure clearly applies also to the 90° block since T_θ is independent of θ.

But the 60° block is evidently pre-eminently suited for an instrument to fulfil these modified requirements, since if we allow such a large tolerance in θ, its measurement to the necessary accuracy can be done quite easily on the scale of the instrument itself, and the only constant left to be determined accurately is the index of the block. By cutting this to 60°, so that these measurements can be made without reference to a standard substance, the instrument can be verified in every essential particular by the user himself.

§ 6. Conclusions.

An effort has been made in the foregoing paragraphs to show that with proper care in the design and subsequent use of the Pulfrich refractometer it will give results accurate to the fifth decimal place, not only in the dispersion, as specially discussed above, but also in the absolute index. It is not suggested that this latter achievement is easy. The mechanical perfection in manufacture which it assumes necessitates the very highest class of workmanship ; but it is not an impossibly high standard, and in view of the important advantages which the method possesses over any absolute method, the production of a really good instrument of the Pulfrich type is of the utmost importance.

With regard to the possibility of a still higher accuracy, it is improbable that absolute indices will ever be obtained closer than 0·00001 by any method of this type on account of the difficulties of measuring prism angles, &c. It should be possible, however, to increase the accuracy of dispersion measurements by increasing the total size of the optical parts in order to permit the use of a telescope of larger aperture and power.

If these were increased to three or four times the present sizes the dispersion measurements might be obtained to two or three units in the sixth place if the micrometer were sufficiently good for the purpose.

The Author would like to express his indebtedness to Mr. T. Smith for his interest in the Paper and for many helpful suggestions ; and to Dr. J. S. Anderson, whose practical experience of the instrument has been of the greatest service in corroborating or modifying the Author's own conclusions.

ABSTRACT.

The Paper deals with points to be observed in the use and design of Pulfrich refractometers. A theoretical investigation of the various errors to which measurements are liable is included.

DISCUSSION.

Mr. F. Simeon expressed his interest in the investigation of the effect of film angle and the use of various liquids to minimise it. Had the author considered the effect of a slight rounding or chamfering of the edge of the specimen on the result ? With regard to the most suitable graticule for the instrument, they had recently experimented with a form which consisted of a rectangular recess in the edge of an opaque screen. The setting was made by making the edge of the coloured band, where it crossed the recess, continuous with the edge of the screen. Except where it crossed the recess, the band was obscured by the screen. This gave very satisfactory settings. He agreed as to the improvement in the sodium setting when the bands were cut down to resolution point. He was interested in the discussion of micrometer errors. They (Messrs. Hilger) had had occasion to go into this carefully in designing their latest instrument. The condition they had adopted was that in which the p error was zero. The tangent error still remained; but as they had made θ_0 half the total range, the error at 5 deg. was only about half what it was in the author's curve. As regards the accuracy in the data for the blocks that makers might be able to supply in future, they had been able to obtain a melting of English glass which in no case showed variations of more than 0·000002 throughout. He had recently made absolute measurements on a prism of right-handed quartz for use as an auxiliary standard, and he was gratified to find that they agreed exactly with the values adopted at the N.P.L., as he had been somewhat uneasy about the discrepancies between his figures and those of other observers. The variation of his individual determinations had been about 0·000014. He observed that no reference to temperature occurred in the Paper. This was by no means a negligible factor in accurate work, especially in the case of quartz and the heavier flint glasses. Had the Author any suggestions as to the best means of temperature control? He suggested that some standard temperature, such as 20°C., should be adopted for the specification of refractive properties.

Mr. T. Smith said that no one reading the Paper could fail to conclude that a very extensive acquaintance with the instrument under all the conditions which could arise in practice, as well as a considerable amount of original investigation, lay behind it. The combination of the facilities for test work and research was essential to the proper consideration of a problem of this kind. This was only possible in an institution where work was carried on on an extensive scale. Unless his memory was wrong, the conclusions in this Paper were based on the experience gained in testing over 2,000 specimens of glass.

Mr. Lamplough (of Chance Bros., Ltd.) said he could confirm the advantage of using different liquids for the different kinds of glass. He had done this for the last six months or so, principally to avoid the shift in the bands produced by vibration and similar causes. He thought the 60 deg. prism would be more difficult to fit with a satisfactory water-jacket than the ordinary type.

Mr. L. C. Martin (communicated remarks) said that it seemed improbable that the types of setting employed in the Pulfrich instrument could ever be so satisfactory as the symmetrical settings possible in the ordinary spectrometer. There were uncertain amounts of irradiation at the retina and aberration in the system which must affect an unsymmetrical setting more adversely. Consistency was no doubt possible; but consistency required to be established between the refractometer and spectrometer indications before the former could be admitted as a standard instrument. We lost the difficulty of providing more than one good surface; but we had to accept certain constants of the instrument which could only be checked by further critical angle methods involving the same possibilities of error. The 60 deg. angle, while involving some loss of sensitiveness, would increase the range of the indications of the instrument, and would permit an interferometer examination of the prism in the required region, and also the determination of its constants on the spectrometer.

The Author, in reply, said that any chamfer on the edge of the specimen was fatal to accuracy, and it was their custom at the laboratory to reject any specimens that were not polished to an absolutely sharp edge. He was interested in Mr. Simeon's description of Hilger's new graticule. It would not, of course, minimise chromatic parallax effects, and seemed, as far as one could judge without experience, to be particularly prone to errors due to irradiation. Further, unless the length of the recess was very short, appreciable error might be introduced due to the curvature of the bands. To make the *p* error zero and $\theta_0 = 2\frac{1}{2}$ deg., as Messrs. Hilger had done, was undoubtedly the best method for a range of 5 deg., but he thought it was a great advantage to increase the micrometer range by an extra 2 deg. He was interested to find the close agreement of the values Mr. Simeon had obtained for quartz and those they had adopted at the laboratory. This certainly increased his confidence in these values. With regard to the question of temperature, the Paper was a somewhat long one, and in order to keep it within reasonable limits he had confined his attention to points that would, if attended to by the user or maker, produce some improvement in the accuracy of the results. This restriction excluded certain points equally vital to the complete treatment of the subject, but which, being inherent in the instrument, were beyond the control either of user or designer; and also those details, such as temperature conditions, which belong equally to all methods of refractometry, and are not, therefore, of special interest in connection with the Pulfrich refractometer. He was glad, however, that some of these points had been raised in the discussion. At present absolute indices were only specified to the fourth decimal place in routine determinations, and to this accuracy the effect of temperature was negligible within the ordinary range encountered in a laboratory. For accurate standardisation work it was necessary to take both temperature and pressure into consideration, since the indices μ_1 and μ_2 occurring in the equations are really the ratios of the indices of the substances to that of air. Thus we had to correct the values of the glass indices in accordance with the temperature coefficients of their indices relative to vacuum,* and apply a further correction for the variation in density of the air, which could be obtained by applying Gladstone and Dale's law. With regard to temperature control, he was convinced that the only accurate method with the Pulfrich instrument was to work at room temperatures. Not only was it impossible by any system of water-jacketing to produce a uniform temperature in a glass block of which one face must necessarily be exposed, but for accurate work it was also necessary to heat the air to the same temperature *right up to the object glass of the telescope.* He thought water-jackets should be dispensed with altogether, as they only tempted the unwary into thinking that temperature coefficients could be measured by their use. The first question raised by Mr. Martin is a point of great importance, though omitted from the Paper on the first of the grounds already explained. When conditions of definition and illumination are good—which they must always be for accurate work—an edge setting of the Pulfrich type can be *repeated* to practically the same accuracy as a symmetrical setting. With regard to irradiation, he had frequently tested the setting with the ordinary cross wires, and it does not vary to an extent that can be detected with bands varying in intensity within the range in which comfortable readings can be made. This being so, whatever personal error there is may be regarded as constant under ordinary good working conditions. As to its magnitude, an absolute test is clearly impossible until an instrument is made sufficiently good that the results are definitely free from all errors not inherent in the method; but there is evidence to show that the personal error is small. As far as our experience goes, settings made by different observers agree to within the limits of sensitivity; whereas when personal errors are large they usually vary considerably with different observers, as in the case, for example, of the unresolved sodium bands cited in the Paper. The author is of the opinion, from these and similar considerations, that the personal error of the setting does not amount to a tenth of a minute. It is interesting to point out, however, that whatever personal error exists is partially compensated in the same way as error in the angle of the block, provided the calibration of μ_2 is made on the instrument itself, because the error in e will be the same for the standardisation measurements as for the subsequent tests. From the curves and equations in the Paper it is easy to calculate that the constant error in e which will affect by 0·00001 the subsequent values of μ_1, when μ_2 is 1·75 is 100 seconds when $\mu_1=1{\cdot}5$; 17 when $\mu_1=1{\cdot}6$, 10 when $\mu_1=1{\cdot}65$ and 6 when $\mu_1=1{\cdot}7$. The compensation in the last case is negligible, but is appreciable for all the more usual glasses. As in the case of the θ compensation, calibration by means of an auxiliary substance is superior in this respect to the absolute method. Thus, for a 90 deg. block calibrated with quartz ($\mu_1=1{\cdot}54$) the equivalent tolerance in personal error is 50 seconds for 1·5, ∞ for 1·54, 1,000 seconds for 1·6, 77 for 1·65 and 25 for 1·7. These figures are to slide rule accuracy only. This would be a really important advantage for the auxiliary substance method of calibration if there was any reason to believe that the personal error with this type of setting amounted to as much as a fifth of a minute.

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* These coefficients can be found for the different types of glass with sufficient accuracy in Hovestadts "Jena Glass."

XV. *On the Accuracy Attainable with Critical Angle Refractometers. By* FREDERICK SIMEON, *B.Sc.* (*From the Research Laboratory of Adam Hilger, Ltd.*)

RECEIVED FEBRUARY 6, 1918.

IT was suggested to the firm by Mr. Guild, of the National Physical Laboratory, that it would be advantageous to substitute a prism of 60° angle for the usual 90° prism used on a Pulfrich refractometer. The present investigation was undertaken in connection with this enquiry, but had been found desirable in dealing also with the Abbe refractometer and its modifications (Butter, Dipping, etc.) made by the firm, in which instruments, too, there is a possibility of using prisms of various angles.

The principle of the method of measuring the refractive index of any substance by observation of the critical angle with respect to it of a prism of higher, and known, refractive index is so convenient, and has found such wide application in instruments for technical and industrial uses, that it has seemed worth while to investigate the accuracy attainable by all instruments employing this method. All such instruments are provided with a prism having two polished faces, in contact with one of which the substance to be measured is placed. Light, which after passing through the substance enters the prism at grazing incidence, emerges from the second face of the prism, and it is this angle of emergence which is the quantity actually determined on the instrument. Various means are used to measure this angle, such as (i.) rotation of an observing telescope attached to a divided circle; (ii.) rotation of the prism by an arm passing over a scale of refractive index; (iii.) movement of the critical edge of the field over a scale in the eyepiece of the observing telescope, etc.; but these differences will be ignored in the present investigation, which concerns the prism system only.

The factors upon which the refractive index (μ) of the substance depends are :—

(*a*) The angle (θ) of the prism,

(*b*) The refractive index (μ_0) of the prism,

(*c*) The angle of emergence (i) of the critical ray from the second surface;

and it is sought to discover with what accuracy these quantities require to be known to ensure a given accuracy in μ; or, conversely, what accuracy in refractive index (μ) is attainable, assuming certain standard errors in these three data.

Consider now the passage of a critical ray through such a refractometer prism, and denote by r_1 and r_2 the angles which the ray inside the prism makes with the first and second surfaces respectively. By the law of refraction (*see* Fig. 1)

$$\mu \sin\frac{\pi}{2} = \mu_0 \sin r_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\sin i = \mu_0 \sin r_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and by geometry

$$r_1 + r_2 = \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eliminating r_1 and r_2 between these three equations, we obtain

$$\mu = \sin\theta\sqrt{\mu_0^2 - \sin^2 i} - \cos\theta \,.\, \sin i. \quad . \quad . \quad . \quad (4)$$

Differentiating (4) with respect to μ

$$1 = [\sqrt{\mu_0^2 - \sin^2 i} \,.\, \cos\theta + \sin i \sin\theta]\frac{\partial\theta}{\partial\mu} + \frac{\mu_0 \sin\theta}{\sqrt{\mu_0^2 - \sin^2 i}} \cdot \frac{\partial\mu_0}{\partial\mu}$$
$$- \cos i\left[\frac{\sin\theta \sin i}{\sqrt{\mu_0^2 - \sin^2 i}} + \cos\theta\right]\frac{\partial i}{\partial\mu}, \quad . \quad . \quad . \quad . \quad (5)$$

or by slightly rearranging the terms

$$1 = \left[\frac{\sin\theta \,.\, \sin i}{\sqrt{\mu_0^2 - \sin^2 i}} + \cos\theta\right]\left[\sqrt{\mu_0^2 - \sin^2 i} \,.\, \frac{\partial\theta}{\partial\mu} - \cos i \,.\, \frac{\partial i}{\partial\mu}\right]$$
$$+ \frac{\mu_0 \sin\theta}{\sqrt{\mu_0^2 - \sin^2 i}} \cdot \frac{\partial\mu_0}{\partial\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It will be noticed that each term of (5) expresses the variation of μ with some one of the three fundamental data of the instrument.

In order to ascertain what accuracy can be relied on with any particular instrument, the value of i for least sensitiveness must be found, and this point will be investigated before the discussion of particular cases of the above expressions.

Assuming a particular prism (*i.e.*, θ and μ_0 constant for the time being), equation (5) reduces to

$$-\frac{\partial\mu}{\partial i} = \cos i\left[\frac{\sin\theta \sin i}{\sqrt{\mu_0^2 - \sin^2 i}} + \cos\theta\right]. \quad . \quad . \quad . \quad (7)$$

Differentiating this with respect to i

$$-\frac{\partial^2\mu}{\partial i^2}=\sin\theta\left[\frac{\sin^2 i\cos^2 i}{(\mu_0{}^2-\sin^2 i)^{3/2}}+\frac{\cos^2 i}{\sqrt{\mu_0{}^2-\sin^2 i}}-\frac{\sin^2 i}{\sqrt{\mu_0{}^2-\sin^2 i}}\right]-\cos\theta\sin i \quad . \; . \; . \; . \; . \quad (8)$$

$=0$ when $-\frac{\partial\mu}{\partial i}$ is a maximum or minimum,

$$i.e.,\ \cos\theta\sin i=\frac{\sin\theta}{\sqrt{\mu_0{}^2-\sin^2 i}}\left[\frac{\sin^2 i\cos^2 i}{\mu_0{}^2-\sin^2 i}+\cos^2 i-\sin^2 i\right]$$

$$\cot\theta\sin i\sqrt{\mu_0{}^2-\sin^2 i}=\frac{\sin^2 i(1-\sin^2 i)+(1-2\sin^2 i)(\mu_0{}^2-\sin^2 i)}{\mu_0{}^2-\sin^2 i}$$

$$\cot^2\theta\sin^2 i(\mu_0{}^2-\sin^2 i)^3=(\sin^4 i-2\mu_0{}^2\sin^2 i+\mu_0{}^2)^2. \quad . \quad (9)$$

This last equation is seen to be a quadratic in $\sin^2 i$, of which there are, in general, two real roots between 0 and +1. Upon substituting the corresponding four values of $\sin i$ into equation (8), it is found that two of these values are not solutions, while a third value does not correspond to a real direction of light in the prism. From a consideration of the change of sign of equation (7), it can be proved that the remaining value of $\sin i$ corresponds to a maximum value of $-\frac{\partial\mu}{\partial i}$—*i.e.*, it gives the value of i for least sensitiveness. It will be noticed that in the case of a 90° prism ($\cot\theta=0$) equation (9) takes a particularly simple form. One root only of the resulting quadratic in $\sin^2 i$ lies between 0 and 1, and only the positive value of the square root of this corresponds to a real direction of light in the prism, this value corresponding again to least sensitiveness. Thus, in cases of practical interest, there is as a rule a unique solution to the problem of determining the position of least sensitiveness.

As special cases of the above formulæ, 90° and 60° prisms of glasses of $\mu_0=1{\cdot}62$ and $\mu_0=1{\cdot}75$ will be considered. The various quantities occurring in equation (5) are given in the following table, in which are also included the various differential coefficients $\left(\frac{\partial\mu}{\partial\theta}\right)_{\mu_0, i}$ denoting the partial differential coefficient of μ with respect to θ, both μ_0 and i being taken as constant, and so on.

μ_0.	1·62.		1·75.	
θ	90·0°	60·0°	90·0°	60·0°
i	48·5°	33·5°	48·0°	32·0°
$\sin\theta$	1·0	0·866	1·0	0·866
$\cos\theta$	0·000	0·500	0·000	0·500
$\cos i$	0·661	0·832	0·667	0·848
$\sin i$	0·748	0·552	0·742	0·529
$\sqrt{\mu_0^2-\sin^2 i}$	1·431	1·524	1·583	1·700
$\dfrac{\mu_0\sin\theta}{\sqrt{\mu_0^2-\sin^2 i}}$	1·131	0·922	1·106	0·892
$\dfrac{\sin\theta\sin i}{\sqrt{\mu_0^2-\sin^2 i}}+\cos\theta$	0·523	0·814	0·469	0·769
$\left(\dfrac{\partial\mu}{\partial\theta}\right)_{\mu_0,i}$	0·748	1·240	0·742	1·306
$\left(\dfrac{\partial\mu}{\partial\mu_0}\right)_{i,\theta}$	1·131	0·922	1·106	0·892
$\left(\dfrac{\partial\mu}{\partial i}\right)_{\theta,\mu_0}$	−0·346	−0·677	−0·313	−0·652

N.B.—The minus sign of the last quantity means, of course, that μ decreases as i increases, i being reckoned positive when on the side of the normal shown in Fig. 1.

Various deductions can be made from these figures. For example, the error in the measurement of refractive indices with a 90° prism owing to uncertainty in the value of the index of the prism itself is slightly greater than the amount of this uncertainty, and slightly less in the case of a 60° prism. Conversely, if the refractive index of the prism is to be determined

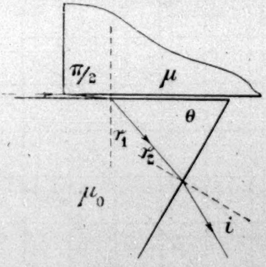

FIG 1.

on the instrument itself by means of auxiliary standards, such as, for example, an accurately measured prism of quartz, then a more accurate value will be obtained in the case of a 90° prism than in that of a 60° prism.

Again, suppose that there is the same uncertainty in the measurement of i whatever the angle of the prism, then readings taken with a 90° prism will be twice as accurate as those taken with a 60° prism.

In order that errors from all these sources may be of the same importance, the angle of the prism must be known to twice the accuracy with which i can be measured. Supposing this last measurement to be made correct to 0·2 minute of arc (which represents about what can be done with the slow motion of the usual form of Pulfrich refractometer), then the angle of the prism should be known to $0{\cdot}1'=6$ seconds. The uncertainty in refractive index of the prism to give a similar error is about 0·00002 in the case of a 90° prism (0·2 minute = 0·00006 radian, approx.), while the corresponding value for a 60° prism is about 0·000045.

The curves in the accompanying diagrams give the variation with i of these differential coefficients for prisms of various refractive indices and angles. They are made to slide-rule

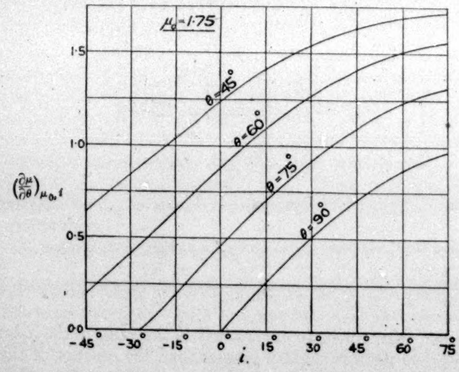

FIG. 2.

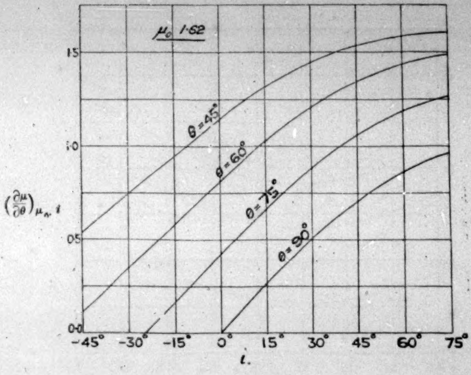

FIG. 3.

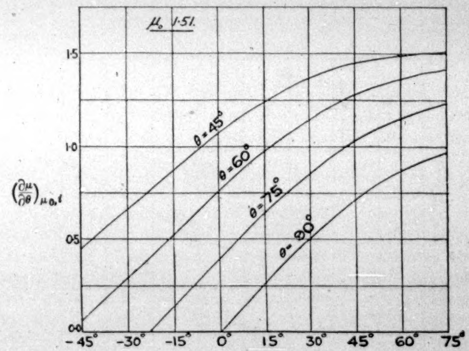

FIG. 4.

Figs. 2, 3 and 4 show the variation of measured refractive index with variation of refractometer prism angle for prisms of refractive indices 1·75, 1·62 and 1·51 respectively.

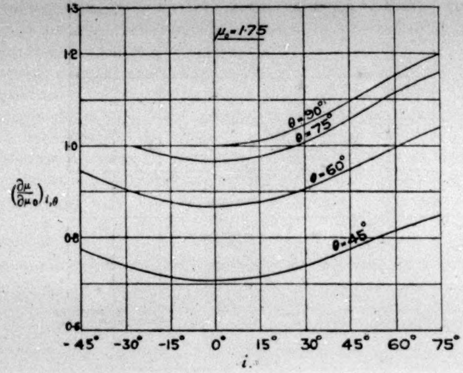
$\mu_0 = 1{\cdot}75$
1·3
1·2
1·0
0·8
0·6
$\left(\frac{\partial\mu}{\partial\mu_0}\right)_{i,\theta}$
θ=90°
θ=75°
θ=60°
θ=45°
-45°
-30°
-15°
0°
15°
30°
45°
60°
75°
i.

FIG. 5.

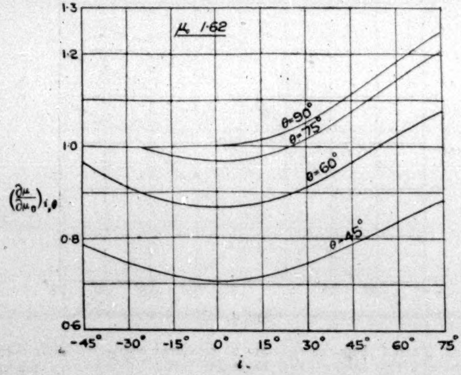
μ_0 1·62
1·3
1·2
1·0
0·8
0·6
$\left(\frac{\partial\mu}{\partial\mu_0}\right)_{i,\theta}$
θ=90°
θ=75°
θ=60°
θ=45°
-45°
-30°
-15°
0°
15°
30°
45°
60°
75°
i.

FIG. 6.

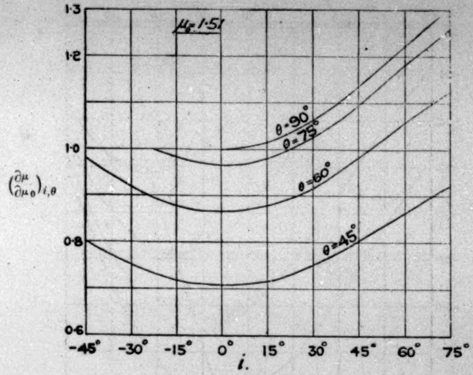

Fig. 7.

Figs. 5, 6 and 7 show the variation of measured refractive index with that of the refractometer prism for prisms of indices 1·75, 1·62 and 1·51 respectively.

accuracy only (as are the rest of the calculations), which is sufficient, as each is multiplied by a small factor in the estimation of errors. The most interesting ones are those connecting i and $\left(\frac{\partial\mu}{\partial i}\right)_{\theta,\mu_0}$.

The locus of the vertices of the curves for any given refractive index (μ_0) is obtained by eliminating θ between equations (7) and (9). The resulting expression is somewhat complicated, but it is seen that the locus is very nearly a straight line, the best straight line being shown dotted in the figures. It will also be seen that this line has very nearly the same slope whatever the value of μ_0. These loci are useful for interpolation of other prism angles.

The above considerations will indicate the nature of the factors controlling the design of prism systems for this class of instrument, having regard to the accuracy of measurement of i which is mechanically attainable.

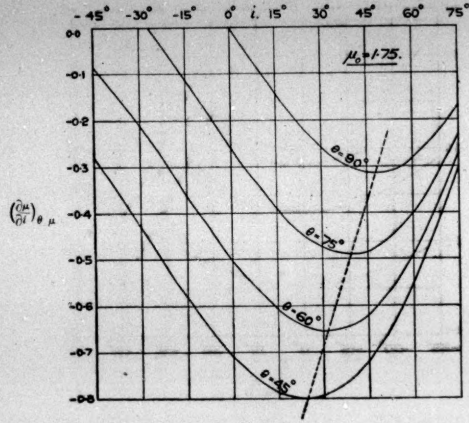
-45°
-30°
-15°
0°
i.
15°
30°
45°
60°
75°
0·0
-0·1
-0·2
-0·3
-0·4
-0·5
-0·6
-0·7
-0·8
$\left(\frac{\partial\mu}{\partial i}\right)_{\theta\ \mu}$
$\mu_0 = 1{\cdot}75.$
θ=90°
θ=75°
θ=60°
θ=45°

FIG. 8.

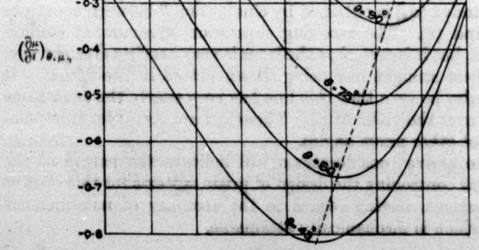
-45°
-30°
-15°
0°
i.
15°
30°
15°
60°
75°
0·0
-0·1
-0·2
-0·3
-0·4
-0·5
-0·6
-0·7
-0·8
$\left(\frac{\partial\mu}{\partial i}\right)_{\theta,\mu_0}$
$\mu_0 = 1{\cdot}62$
θ=90°
θ=75°
θ=60°
θ=45°

FIG. 9.

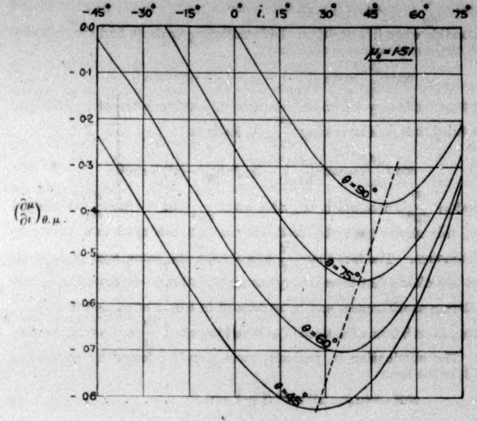

Fig. 10.

Figs. 8, 9 and 10 show the variation of measured refractive index with variation of angle of emergence of the critical ray from prisms of refractive indices 1·75, 1·62 and 1·51 respectively.

ABSTRACT.

The three factors controlling the determination of a refractive index by means of a critical angle refractometer are, so far as the prism system is concerned, (i.) the angle of the prism, (ii.) its refractive index, and (iii.) the angle of emergence of the critical ray from the second prism face. Expressions are obtained for the variation of the required refractive index with each of these factors separately and curves are given connecting these variations with the angle of emergence from the second prism face for various prism angles.

DISCUSSION.

Mr. T. Smith thought it was unfortunate that the Author had plotted his curves against angles of emergence. What one wanted to know in practice were the values of these functions when determining a particular refractive index. The curves should, therefore, have been plotted against μ. As they appeared in the Paper they were misleading. The $\partial\mu/\partial\theta$ graphs, for instance, conveyed the impression that this quantity depended on θ, whereas in reality it depended only on μ_0 and μ, and had nothing to do with the angle of the prism. The mathematics could have been simplified somewhat. The investigation of the sensitiveness may conveniently start from the equation

$$\sin i=(\mu_0^2-\mu^2)^{\frac{1}{2}}\sin\theta-\mu\cos\theta, \qquad (1)$$

so that

$$-\cos i\frac{di}{du}=\mu(\mu_0^2-\mu^2)^{-\frac{1}{2}}\sin\theta+\cos\theta \qquad (2)$$

showing that $\frac{di}{d\mu}$ is necessarily negative, since refraction in the way considered is only possible when θ is positive. Differentiating again to find the positions in which the sensitiveness is stationary,

$$\cos i\frac{d^2i}{d\mu^2}=\sin i\left(\frac{di}{d\mu}\right)-\mu_0^2(\mu_0^2-\mu^2)^{-\frac{3}{2}}\sin\theta, \quad . \quad . \quad . \quad (3)$$

showing that stationary values only occur with positive values of i. Differentiating a third time, and assuming $\frac{d^2i}{d\mu^2}=0$, leads to

$$\cos i\frac{d^3i}{d\mu^3}=\cos i\left(\frac{di}{d\mu}\right)^3-3\mu_0^2\mu(\mu_0^2-\mu^2)^{-\frac{5}{2}}\sin\theta$$

and therefore $\frac{d^3i}{d\mu^3}$ is necessarily negative when $\frac{d^2i}{d\mu^2}=0$. Since $\frac{di}{d\mu}$ is always negative, this shows that the stationary values are minimum values in absolute amount. The fact that $\frac{d^3i}{d\mu^3}$ has always the same sign is in itself sufficient to indicate that there is only one real solution to the equation $\frac{d^2i}{d\mu^2}=0$. The minimum sensitiveness can be expressed in terms of μ_0 and μ only by eliminating i and θ from equations (1), (2) and (3) when $\frac{d^2i}{d\mu^2}$ in (3) is equated to to zero. The elimination is, perhaps, most simply effected by combining (1) and (2) in the form

$$\sin^2 i+(\mu_0^2-\mu^2)\cos^2 i\left(\frac{di}{d\mu}\right)^2=\mu_0^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Differentiating this with respect to μ, removing the common factor $2\cos i\frac{di}{d\mu}$, and putting $\frac{d^2i}{d\mu^2}=0$, gives

$$\left\{(\mu_0^2-\mu^2)\left(\frac{di}{d\mu}\right)^2-1\right\}\sin i=-\mu\frac{di}{d\mu}\cos i \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Introduce the factor $\sin^2 i+\cos^2 i$ on the right of (4), and eliminate i between (4) and (5). The result is obviously

$$(\mu_0^2-\mu^2)(x-1)^2(x-\mu_0^2)=\mu^2(\mu_0^2-1)x \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where x is written for $(\mu_0^2-\mu^2)\left(\frac{di}{d\mu}\right)^2$. The only roots of (6) which give a real solution to the problem are derived from positive values of x. The form of (6) shows that all positive values of x must exceed μ_0^2. If μ_0^2+y were written for x, the terms in y^3 and y^2 would evidently have the same sign, and would differ in sign from the term without y. There is thus only one positive root of (6). The curve giving the corresponding value of $\frac{di}{d\mu}$ will pass through the lowest points of the curves A, B, C, D of Fig. 7 in Guild's Paper.* The value of θ with which these values of $\frac{di}{d\mu}$ are associated may be found from

$$\tan\theta=\frac{\mu(\mu_0^2-\mu^2)^{\frac{1}{2}}x}{(\mu_0^2-\mu^2)x-\mu_0^2}.$$

There is one further point of importance to which attention may be called. The minimum values of the sensitiveness given by the above formulæ are not absolute minima in the sense that higher values will be obtained by using a standard block of any other angle than that indicated. For in all cases

$$\frac{di}{d\theta}+(\mu_0^2-\mu^2)^{\frac{1}{2}}\frac{di}{d\mu}=0.$$

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* This Number, page 157.

Differentiating this with respect to μ leads to

$$\frac{d}{d\theta}\left(\frac{di}{d\mu}\right)=\mu(\mu_0^2-\mu^2)^{-\frac{1}{2}}\frac{di}{d\mu}$$

if $\frac{d^2i}{d\mu^2}=0$. Thus the sensitiveness determined by the condition $\frac{d^2i}{d\mu^2}=0$ is increased by increasing θ and decreased by reducing θ. Minimum sensitiveness in the absolute sense is given by the condition

$$(\mu_0^2-\mu^2)^{\frac{1}{2}}\frac{d}{d\theta}\left(\frac{di}{d\mu}\right)=\mu\frac{di}{d\mu}-(\mu_0^2-\mu^2)\frac{d^2i}{d\mu^2}=0,$$

which by the differential of (4) reduces to $\sin i=0$, or $(\mu_0^2-\mu^2)\left(\frac{di}{d\mu}\right)^2=1$. Equation (4) shows that the latter solution is impossible. The minimum sensitiveness is given by

$$-\frac{di}{d\mu}=\mu_0(\mu_0^2-\mu^2)^{-\frac{1}{2}}$$

on a block whose angle θ satifies $\mu_0\sin\theta=\mu$ or $\sec\theta=-\frac{di}{d\mu}$. This result shows that Guild's curves A, B, C, D (Fig. 7) are touched by their envelope where the emergent ray is normal to the second face of the glass block. The same conclusions obviously follow from equation (4) without the need of differentiating with respect to θ.

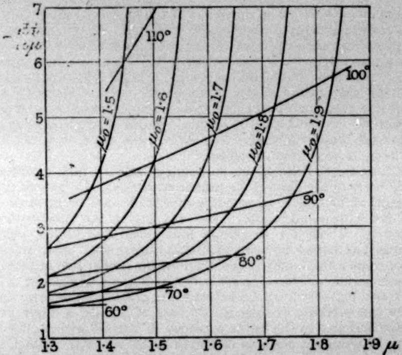

Diagram showing minimum values of the sensitiveness with two given kinds of glass, and the angles of the block to which they relate.

Example of use of diagram.—With a standard block of 80° angle and refractive index 1·9 the minimum sensitiveness is 2·5 (or 5 seconds for a fifth place unit in the index) and occurs for a specimen of approximate refractive index 1·64.

Mr. LAMPLOUGH said he had not used the new setting described by Mr. Simeon sufficiently to vouch for its accuracy, but it was certainly a very comfortable type of setting to use. He would like to know the purpose of serrating the edge of the shutter provided both in the Zeiss and Hilger instruments. What was the purpose in Messrs. Hilger's model of reversing the parts of the instrument right for

Mr. J. **Guild** thought that the Author, in finding the angle of emergence for minimum sensitivity, and comparing the liability to various errors, in these circumstances was comparing non-comparable quantities. It was true that the least sensitivity encountered in the working range of the instrument was an important factor in determining the relative advantages of different types; but the sensitivity minimum for the acute angled blocks occurred for values of μ quite outside the range for which blocks of that index would be used; in most cases they were even outside the range of indices found in solid or liquid substances. Thus the tabulated comparison of 60 and 90 deg. blocks was not valid, for the 60 deg. block ($\mu_0=1\cdot75$) was functioning under the condition $\mu=1\cdot18$, as against $\mu=1\cdot58$ in the case of the 90 deg. The corresponding figures for $\mu_0=1\cdot62$ were 1·04 and 1·43. Quite erroneous conclusions would be drawn from the tables. For example, one would gather that $\delta\mu/\delta\theta$ was much larger with 60 deg. blocks than with 90 deg. whereas this coefficient was independent of the angle, depending only on μ_0 and μ. The conclusions drawn by the Author were unduly in favour of the 90 deg. block for this reason. The curves as at present plotted required subsidiary calculation in order to obtain the information usually required.

Mr. J. W. **French** (of Messrs. Barr & Stroud); communicated remarks. The Author has confined himself to questions involved in the first design of the refractometer—that is, as regards the choice of the best prism angle. It would be very useful if he would extend the Paper to cover factors that determine the accuracy obtainable in the actual use of these instruments. These factors include: (1) The sharpness of the angle of the specimen; (2) the flatness of the contact surface of the specimen; (3) the flatness of the surface of the refractometer prism; (4) the accuracy with which the specimen is put into contact with the refractometer prism; (5) variations in the illumination; (6) errors in the quartz calibration prism; (7) temperature of the specimen; (8) temperature of the refractometer prism. Items (2) and (3) are of particular importance, considering the oblique transmission of the rays through a possibly curved surface. If these various items could be expressed in the same clear manner as those dealt with in the Paper it would be very helpful to users of such refractometers.

Mr. **Churcher** (of Bellingham & Stanley) said it had recently been observed by users of the Zeiss-Abbe refractometer that the instrument fails, owing to want of illumination, when measurements are required of liquids having an index exceeding 1·52. This was due to the substitution of a crown prism ($n_D=1\cdot52$) for the dense flint prism ($n_D=1\cdot75$), which was apparently used at first as the lower or illuminating prism. As is well known, in using the Abbe refractometer, one observes the border line of total reflection of a transparent substance in contact with a dense flint prism of known refractive index, light entering the substance at grazing incidence. In order to facilitate the entrance of light in the case of liquids, a lower prism of similar shape and size to the upper is placed in contact with it, the film of liquid being between the surfaces of the two prisms. The contact surface of the lower prism is left unpolished, and is intended to act as a scattering surface. If the scattering action were perfect, the refractive index of the lower prism would be immaterial. In the process of cleaning the ground glass surface, the sharp points are removed while the depressions remain. Such a surface approximates to a polished surface, especially at oblique illumination. Thus, if the index of the lower prism is less than that of the liquid film, the bulk of the rays entering the film are bent towards the normal at the interface, and cannot strike the second prism surface at grazing incidence. It is only when the lower prism is of higher index than the liquid that the light, being bent away from the normal, can enter the film in any quantity in a direction parallel to the interface so as to strike the second surface at grazing incidence. The Zeiss model was awkward to use with solid and plastic bodies such as glass, resins, gums, &c., as the prism box is designed to open towards the operator and away from the source of light. It is necessary in these cases to turn the instrument round, relying for illumination on the light reflected from the grey surface of the lower prism and to read the scale backwards. It is plainly stated in the Zeiss catalogue that the Abbe refractometer may be used for the measurement of the refractive indices of liquids from 1·30 to 1·7, and that the liquid to be examined is enclosed between two flint glass prisms. As neither of these statements is correct, it shows that manufacturers taking up the supply of instruments in the interests of British trade would do well to avoid the tendency to accept the optical and mechanical design of German instruments as incapable of improvement. It would be preferable to consult users of the instruments, and from accumulated experience to introduce entirely new designs, embodying as far as possible any suggestions which increase the utility and ease of manipulation of the apparatus.

The AUTHOR, in reply, said: Mr. Guild's remark concerning the subsidiary calculation necessary to determine the refractive index for which the sensitiveness is a minimum was quite justified. Nevertheless, the angle of emergence was a quantity which had been found useful in considering points in the design of refractometers. In reply to Mr. Lamplough's questions, the purpose of the serrated edge to the shutter was to enable distinction to be made in the two edges of a coloured band in the field of view, the unimportant edge only being affected by interposition of the shutter. It was not, of course, necessary for a practised observer. The purpose of reversing the parts of the instrument in Messrs. Hilger's new model was to bring all screw heads, &c., within reach of the observer's right hand. Additional support had been given to the various attachments, and a greater useful range of movement of the telescope provided. With regard to Mr. French's remarks, the Paper only attempts to deal with points in the first design of such instruments. The other factors named are of great importance in practice, and mention of several of them has been made in connection with the preceding Paper, though a fuller discussion is undoubtedly desirable.

XVI. *On the Times of Sudden Commencement of Magnetic Storms.* *By* S. CHAPMAN.

COMMUNICATED BY DR. C. CHREE, F.R.S.

RECEIVED MARCH 22, 1918, AND, IN REVISED FORM, APRIL 13, 1918.

§ 1. As at first communicated, this Paper was written in April 1917, or thereabouts. Some revision has seemed desirable, partly in view of a subsequent change in my ideas concerning the nature of magnetic storms, and partly because a discussion of the same data by Dr. Chree,* with which I was not then acquainted, rendered certain introductory matter superfluous.

In 1910 a Paper by Dr. Bauer† renewed interest in the question as to how far magnetic storms commence simultaneously at different parts of the earth. W. G. Adams and W. Ellis‡ had shown, long before, that the commencement-time is nearly the same, in great magnetic storms, over a large part of the earth; as regards intervals of time amounting to a few minutes, however, the question was still open. Dr. Bauer claimed in his Paper to have demonstrated that the time of propagation "round" the earth was actually of the order of 3 or 4 minutes, and he indicated the supposed direction and speed of propagation in the cases of several particular storms. These conclusions excited some controversy, being contested by various authors.§ In order to assist in settling the point, Dr. Bauer collected, with the aid of the directors of 32 contributing observatories, new data for the commencement-times of 15 magnetic storms.|| This set of material is the amplest and most suitable which has so far been available for the discussion of the subject, and there can be no doubt that its compilation has increased the interest and care taken in the accurate time-scaling of magnetograph registers.

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* C. Chree, "Proc." Phys. Soc., XXVI., p. 137, 1914; also *ibid.*, XXIII., p. 49, 1910; and "Nature," LXXXVI., p. 79, 1911.

† L. A. Bauer, "Terr. Mag.," XV., p. 9 (also pp. 107, 221), 1910; "Nature," LXXXV., p. 306, LXXXVI., p. 9, 1911.

‡ W. G. Adams, "Brit. Ass. Rep.," 1880, p. 201, 1881, p. 463; W. Ellis, "Nature," XXIII., p. 33, 1880, "Proc." Roy. Soc., LII., p. 191, 1892; also van Bemmelen, "Amst. Acad. Sci.," Versl., XV., p. 250, 1906, Proc., IX., p. 266, 1906, Proc., 1908; also Birkeland, Norwegian Aur. Pol. Exp., 1902-3, I., p. 63, 1908.

§ Faris, "Terr. Mag.," XV., pp. 93, 213, 1910, and XVI., p. 109, 1911; Chree, *l.c.*; Krogness, "Nature," LXXXV., p. 170, 1910; Walker, *ibid.*, p. 236, 1910; van Dijk, *ibid.*, LXXXVI., p. 44, 1911; Birkeland, *ibid.*, LXXXVI., p. 79, 1911.

|| "Terr. Mag.," XVI., pp. 85, 163, 1911.

Up to this point the discussion seemed to have terminated without bringing about agreement between the opposing sides. The verdict of Dr. Chree and Dr. Krogness was that Dr. Bauer's observational conclusions were not established, so that also the theory which he based upon them could not be maintained.

No discussion of the subsequent collection of data has appeared in "Terrestrial Magnetism," and at the time of my own examination of it I was unaware that the material had already been discussed by Dr. Angenheister[*] and Dr. Chree.[†] So far as regards the above-mentioned theory, the three discussions agree in concluding that the new data afford no evidence in its favour, but rather to the contrary. In view, especially, of Dr. Chree's very complete treatment it would be useless to labour this point further; as regards the discussion of instrumental points, and the grouping of the time-data according to latitude and longitude, a general reference to his Paper will suffice. The justification for bringing forward this further discussion must rest upon the novelty of the point of view underlying it, which is different from that adopted by either Dr. Angenheister or Dr. Chree. Although no definite positive results are here derived from the existing data, the method of treatment does, I think, give promise of such when more material is forthcoming.

§ 2. In the various Papers cited only three possibilities seem to have been contemplated regarding the commencement-times. Thus, Dr. Chree[‡] remarks: "As regards these 'sudden' changes three things are conceivable: they may be absolutely simultaneous at different stations; there may be a very small difference of time, corresponding to the rate of propagation of electromagnetic waves; or, finally, there may be, as Dr. Bauer concludes, longer intervals, amounting to several minutes, for stations remote from one another. Under existing conditions of registration one cannot decide between the first two possibilities. As between these two and the third, a decision should not be impossible."

While reading Mr. Maunder's Papers[§] on the recurrences of magnetic storms at intervals equal to the synodic rotation-

[*] Angenheister, "Nach. d. K. Ges. d. Wiss. zu Göttingen, Math.-Phys. Kl.," 1913.

[†] Chree, *l.c.* (1914).

[‡] C. Chree, Proc. Phys. Soc., XXIII., p. 49, 1910.

[§] E. W. Maunder, "M.N.R.A.S.," Nov. 1904, April, May, 1905, Nov. 1915; "Journ. Brit. Ast. Ass.," XVI., p. 140, 1906; "Astrophysical Journ.," XXI., p. 101, 1905.

period of the sun (27·3 days), a fourth possibility naturally suggested itself. Mr. Maunder's results suggest that storms are occasioned by some solar agent which is transmitted along narrow, well-defined streams, issuing from and rotating with the same period as the sun. The direction of solar rotation is such that the streams, if suitably situated, will overtake the earth on the afternoon or P.M. side.* Further, since the earth's angular diameter, viewed from the sun, is only 18″·6, it is easy to calculate that the time necessary for a stream to sweep right across the globe is almost exactly 30 seconds.

The above considerations suffice to explain how, without regard to any particular theory as to the precise mechanism of magnetic storms, the presumption arose that the relative time of commencement of a storm, at different stations, depends mainly on the orientation of the latter, at the time, relative to the sun. The determining factor, it is thus suggested, is neither mainly latitude, nor "absolute" longitude, but principally longitude relative to the sun. Now this relative longitude depends simply on the local time† at the station at the commencement of the storm. The local times of commencement, therefore, form the basis of classification of the data in the present discussion.

§ 3. Again, the *order of magnitude* of the time-differences to be expected, between the commencements at different stations, is suggested by the time (30 seconds) taken for a stream to sweep across the earth. The differences, therefore, are to be reckoned in fractions of a minute, or in seconds, and not, as on the second and third of the hypotheses mentioned by Dr. Chree (§ 2) in small fractions of a second, or in whole minutes. It is true that an electromagnetic change occurring at any station will produce *some* effect, propagated with the speed of light, at all other stations on the earth, but the magnitude of the effect diminishes so rapidly with increasing distance from the source that it soon becomes inappreciable. The initial impulse of a magnetic storm, like the diurnal magnetic variations, is mainly produced by atmospheric currents situated in the comparatively near neighbourhood of each station. The time differences of commencement at different stations are due to the lack of simultaneity of arrival of the solar agent of excitation at those places.

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* If we imagine ourselves looking from the sun towards the earth, the latter will be seen to travel along its orbit from right to left. The solar streamers rotate in the same direction, but, moving much more rapidly, overtake the earth. The earth and sun revolve in the same direction, so that the hemisphere visible from the sun (the "day" hemisphere) will be turning from left to right. The local time for all stations to the right of the terrestrial meridian plane containing the sun will be *after* noon, so that the right-hand hemisphere will be termed the P.M hemisphere (*post meridiem*). In this paper it will be assumed, for simplicity, that the axes of earth and sun are perpendicular to the ecliptic, or plane of the earth's orbit. The earth will also be regarded as divided into four quadrants relative to the position of the sun at the time, the local time in the quadrants, which we may respectively term the night A.M., the day A.M., the day P.M., and the night P.M., ranging from midnight to 6 a.m., 6 a.m. to midday, midday to 6 p.m., and from 6 p.m. to midnight.

† Local time measures the longitude, at the rate of 15° per hour, from the "midnight" meridian, which is in the plane containing the sun, but on the dark side of the earth.

Since this discussion was first completed, having carefully studied the nature of magnetic storms, my then vague ideas have given place to fairly definite conclusions regarding their origin and mechanism.* These do not enable me, however, to conclude any more definitely than at first whether the *precise* range of commencement-time to be expected, in the occurrence of a storm at stations distributed over the earth, is likely to be rather more or rather less than the above 30-second interval. The electric corpuscles which compose the streams do not impinge directly upon the atmosphere, but are deflected by the earth's magnetic field, while still at distances comparable with the earth's diameter. Prof. Störmer † has made calculations as to the paths of single electric corpuscles in the earth's field, and has found that the variety of possible types is very great. His calculations appear to be too ideally simplified, as yet, to give results accurately representative of the actual conditions, the mathematical difficulties even in the simple case of one electron being very formidable. The magnetic storms themselves indicate that the particles are precipitated more upon the P.M. than upon the A.M. hemisphere,‡ and as the former hemisphere is also the one on the side of approach of the streams, it is not unlikely that the initial impulse of a storm occurs first in that hemisphere. The discussion of the data of this Paper seems to favour this view, though the evidence is too slight to warrant a definite conclusion. It may be said, therefore, that at present theory gives but little guide as to the precise results to be expected, and that more observational material is required in order to ascertain the actual facts.

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* These conclusions are described in a Paper communicated to the Royal Society on April 10, 1918.

† Störmer, "Arch. d. sc. phys. et nat. Genève," 1907, and various articles in "Terrestrial Magnetism" and elsewhere.

‡ Cf. the footnote to p. 207; also the Paper on magnetic storms referred to in the last footnote but one.

§4. The data to be considered are confined to the times of commencement as measured from the horizontal force record alone. The sudden perturbation at the beginning of a storm is generally most pronounced, and therefore most accurately measurable, in this element. The principal source of error in the data appears to be the uncertainty as to which is the precise movement on the registers which marks the sudden commencement. If proper precautions are taken, the time of a given movement on the record is measurable with an accuracy of about a minute. An inspection of the data suggested that (with perhaps one exception) the *systematic* error of time determination, at any of the 32 observatories which contributed data, did not exceed one minute. I therefore decided to reject any observation which stood out from the mean time of commencement for the particular storm by two minutes or more, on the ground that the difference probably arose from a choice of the wrong movement for measurement. Some observatories gave more than one commencement-time for one or two of the storms, accompanied by drawings showing the magnetic movements to which the times corresponded. In these cases it was usually possible to choose, with little uncertainty, the precise movement which had been taken by the majority of the other observatories. The total number of rejected values was not large,* while the number retained was 342; if all the 32 observatories had given data for each storm, the total number of values would have been 480, but some observatories did not furnish complete data. The table (facing p. 212) shows the rejected values as well as those retained.

The mean time of commencement for each storm was determined from the values given by the several observatories, excluding those which were rejected. The residual differences, "actual value—mean," are given in the table, for all the contributing observatories. Where a query is attached to an observation, this was done by the observatory which supplied it, and these values have in all cases been bracketted. A few other particulars regarding the table are indicated in the notes appended to it. The observatories are arranged in order of longitude, and the storms in order of date. Columns are added giving the systematic difference of time from the mean, and also the mean numerical residual, for each *observatory*; in a row beneath, the mean numerical residual for each *storm* is given. The number of residuals concerned in each mean is indicated by a suffix. The unit in which all these* quantities are expressed is one-tenth of a minute.

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* Counting those rejected on ground of magnitude alone, there were 31 negative and 22 positive residuals bracketted.

For some storms the data are clearly much better than for others; this is probably owing to the greater definiteness of the initial perturbation in the former cases. Magnetic storms show marked differences in this particular. Storm 1 is an example of a "good" storm—only two values are rejected out of 27, and the mean numerical residual of the other 25 is 0·4 minute. The numerical mean residual derived from the 342 values all taken together is about 0·7 or 0·8 minute. This should allow a clear indication to be given of times of the order of two or three minutes, such as were suggested by Dr. Bauer; this, it will be seen, is not shown, the residuals, which are of about one minute in magnitude, largely consisting of accidental error. When it is remembered that, with the ordinary time-scale, 1 mm. on a magnetograph sheet represents about three minutes of time, this degree of accidental error is not surprising.

§ 5. In accordance with the ideas described in §§ 2, 3, the residuals for each storm were divided up into four groups, according to which of the four quadrants mentioned in the footnote to § 2 the corresponding station was situated in at the time of commencement. This was done by considering the longitudes, west of Greenwich, of the four principal meridians with respect to the sun, viz., the midnight, midday, 6 a.m. and 6 p.m. (local time) meridians. Thus, in storm 1, the commencement occurred at 19h 56·7m G.M.T., so that the midnight meridian was this amount (which in arc is 299°·2) to the west of Greenwich. The longitudes of the four meridians (0, 6, 12 and 18 hours local time) were therefore 299°·2, 209°·2, 119°·2, and 29°·2. In the table dividing lines representing these longitudes were drawn for the first storm (and in a similar fashion for the other storms), these lines being respectively light and continuous, broken, thick and continuous, and broken. The stations in the night A.M. quadrant at the commencement of any storm are consequently those above the light continuous line and below the consecutive (broken) line. In this way it becomes an easy matter to group the residuals in the table according to their corresponding quadrant or hemisphere.

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* At the foot of the table, however, are other quantities, described in § 5, which are expressed in units of 1 second.

At the foot of the table rows are to be found which give for each storm the means of the residuals in the four hemispheres, day, night, P.M. and A.M., and also the differences between the results for complementary hemispheres. These values are all given in units of 1 second, not 0·1 minute as in the remainder of the table. The mean difference for either pair of hemispheres in the case of no storm exceeds 60 seconds, and these mean differences for individual storms are clearly still affected by considerable accidental error. Any systematic quantities underlying them must, therefore, be of the order of 20 seconds or less. Taking all the residuals in any one quadrant together, we obtain the following results. The number of residuals, their sum* (in minutes) and their mean (in seconds), are given in order :—

Night P.M. quadrant (84): Sum −5·0m., Mean −3·6s.
Day P.M. „ (89): „ −3·8m., „ −2·6s.
Night A.M. „ (87): „ −1·1m., „ −0·8s.
Day A.M. „ (82): „ +7·2m., „ +5·3s.

The quadrants are arranged in the numerically increasing order of their mean residuals. This order is not to be regarded as the true order of arrival of the initial impulse of a magnetic storm at the different quadrants; more data would be needed to establish any such conclusion. There is perhaps enough evidence to favour the opinion that the P.M. hemisphere is the one first affected by the storm, though even this is doubtful. If we group the quadrants into pairs of hemispheres, we obtain the following results :—

P.M. residuals (173): Sum −8·8m.: Mean −3·2s. } P.M.-A.M., −5·3s.
A.M. „ (169): „ +6·1m.: „ +2·2s.

Day residuals (171): Sum +3·4m.: Mean +1·2s. } Night−day, −3·3s.
Night „ (171): „ −6·1m.: „ −2·1s.

The latter result is certainly more unlikely than the former, but cannot be dismissed as (theoretically) incredible. In order to test how far accidental error might account for these small differences of time, I made the following two quite arbitrary subdivisions of the data :—

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* It may be noted that the sum of all the residuals is −2·7m., which arises from an accumulation of small errors, less than 0·1m., in the adopted mean times of commencement of the storms.

First 16 observatories (172 residuals) :
Sum −1·9m. : Mean −0·7s.
Second 16 observations (170 residuals) :
Sum −0·8m. : Mean −0·3s.
Difference 0·4s.

Upper right-hand diagonal half (177 residuals) :
Sum +2·5m. : Mean +0·8s.
Lower left-hand diagonal half (165 residuals) :
Sum −5·2m. : Mean −1·9s.
Difference 2·7s.

The latter, which is about equal to the difference between the day and night residuals, confirms the unlikelihood of the latter being real. The matter for surprise in all cases, however, is the extreme smallness of the residuals, and this suggests that it should not be very difficult, even with present instrumental means, to determine any systematic time differences between the different quadrants, if these amount to so much as 10 seconds. The extreme range (30 seconds, more or less) would not, of course, show in the mean of whole quadrants or hemispheres ; and the fact that the majority of observatories contributing data are in high latitudes likewise tends to minimise the differences to be expected.

ABSTRACT.

The Paper is a discussion from a new standpoint of the data, collected by Dr. Bauer, for 15 magnetic storms. Maunder's work on the recurrence of magnetic storms at intervals equal to the rotation period of the sun suggests that storms are due to some solar agent transmitted along narrow well-defined streams, issuing from and rotating with the sun. This suggests the view that the relative time of commencement of a storm at different stations depends mainly on the orientation of the latter at the time, relative to the sun, *i.e.* on the local time at the station. This forms the basis of the classification in the Paper.

DISCUSSION.

Dr. Chree said that after hearing the Paper he was uncertain whether Dr. Chapman did or did not believe that the figures *proved* anything. He (Dr. Chree) did not think that they did. To deal with a question of a few seconds' difference at different stations, it would, he thought, be desirable to employ not merely a more open time scale, but magnetographs of greater and desirably uniform sensitiveness. When a magnetic change became recognisable depended on the size of the movement and the sensitiveness of the magnetograph. An apparent difference in time between day and night hemispheres would naturally arise if movements tended to be larger in the one than the other. One objection which he had urged against Dr. Bauer's views also applied to the hypothesis considered by Dr. Chapman, viz., that as soon as a disturbance began at any part of the earth's atmosphere it would naturally be propagated b

Observatory.	West longitude.	Latitude.	Index number of storm, with date and mean time of commencement.															Mean residual.	Mean numerical residual.
			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15		
			1906, July 29. $19^{h}56{\cdot}7^{m}$	1906, Aug. 7. $13^{h}38\ 1^{m}$	1906, Dec. 21. $21^{h}30{\cdot}6^{m}$	1907, Feb. 9. $14^{h}13{\cdot}0^{m}$	1907, July 10. $14^{h}22{\cdot}7^{m}$	1907, Oct. 13. $7^{h}43{\cdot}2^{m}$	1908, Mar. 26. $17^{h}41{\cdot}4^{m}$	1908, Aug. 19. $0^{h}14{\cdot}5^{m}$	1908, Sep. 11. $7^{h}\ 20{\cdot}7^{m}$	1908, Sep. 11. $21^{h}46{\cdot}6^{m}$	1908, Sep. 28. $8^{h}41{\cdot}9^{m}$	1908, Sep. 29. $1^{h}32{\cdot}1^{m}$	1909, May 14. $4^{h}55{\cdot}0^{m}$	1909, Sep. 25. $8^{h}38{\cdot}2^{m}$	1909, Sep. 25. $11^{h}40{\cdot}7^{m}$		
	°	°																	
Greenwich*	0·0	51·5 N	3	(−37)	−6	10	3	8	(16)‡	5	−7	4	1	9	(30)	18	−7	3_{12}	7
Kew*	0·3	51·5 „	3	9	4	0	−7	−2	(16)‡	5	(−27)	4	1	−1	(20)	0	3	2_{12}	3
Stonyhurst	2·5	53·8 „	−10	−6	(−25)	(−21)	−2	−5	...	(−29)	(−29)	−14	−7	19	−7	§	...	-4_{8}	9
Eskdalemuir	3·2	55·3 „	...	...	...	...	...	...	...	...	...	...	...	...	...	§	8	8_{1}	8
Falmouth*	5·1	50·1 „	3	−1	(34)	(20)	13	18	(26)	5	8	−11	1	−11	0	(−22)	−17	1_{11}	8
Pilar	63·9	31·7 S	...	...	−5	(25)	9	−6	11	−15	(−36)	(−22)	(−29)	(−34)	(−26)	2	6	0_{7}	8
Porto Rico	65·4	18·1 N	−3	−14	(34)	1	−1	2	0	7	−2	6	−11	−3	−8	4	1	-2_{14}	5
Cheltenham	76·8	38·7 „	−2	9	12	2	0	−2	(36?)	−1	−6	1	−6	−9	14	8	5	2_{14}	5
Agincourt	79·4	43·7 „	−4	(−20)	4	−5	−2	0	14	−3	4	11	−1	−18	(23)	5	7	1_{13}	6
Baldwin	95·2	38·8 „	4	11	0	0	5	1	8	4	1	(25?)	−3	−5	−3	−5	−2	1_{14}	4
Sitka	135·3	57·0 „	−7	(21)	0	5	−3	(8?)	...	12	3	2	10	...	(21)	16	3	4_{10}	6
Honolulu	158·1	21·3 „	(−24)	−9	(−23)	−10	−18	−11	−12	−7	9	−6	12	−1	−2	−3	−1	-5_{13}	8
Samoa	171·8	13·8 S	2	−5	(−31)	−2	0	6	6	−2	−1	11	−6	−5	...	...	...	0_{11}	5
Zikawei*	238·6	31·2 N	(−27)	−11	−6	(−20)	−7	8	......	5	−17	−16	(−29)	9	−10	−12	−7	-6_{11}	10
Batavia	253·2	6·2 S	−6	−18	−6	−10	−4	(−24)	(−30)	2	−18	5	7	(27)	2	5	19	-2_{12}	8
Toungoo	263·5	18·9 N	6	−3	−5	7	18	−3	−4	...	...	...	...	...	...	−8	−5	0_{9}	7
Barrackpore	271·6	22·8 „	3	−4	7	...	7	−3	1	−4	...	11	−6	18	...	−4	−5	2_{12}	6
Dehra Dun	281·9	30·3 „	2	−2	9	12	−3	4	−5	−1	(21)	5	3	−2	...	−12	−2	1_{13}	5
Kodaikanal	282·5	10·2 „	11	−2	5	18	(20)	−1	1	−1	(25)	(20)	3	4	...	−7	12	4_{11}	5
Bombay	287·2	18·9 „	−3	1	−4	1	−3	7	0	−13	0	−12	−5	−9	−13	−2	15	-3_{15}	6
Ekaterinburg	299·4	56·8 „	0	7	9	10	6	(24)	(21)	0	...	...	9	−3	16	6	−7	5_{11}	7
Mauritius	302·4	20·1 S	−7	19	−16	†	†	†	†	†	†	†	†	†	†	†	†	-1_{3}	14
Helwan*	328·7	29·9 N	...	...	...	...	−17	8	6	15	13	(24)	(−29?)	9	−10	−2	−17	0_{9}	11
Pola	346·2	44·9 „	9	(−21)	(−20)	−11	(24)	(−29)	−16	−9	...	†	−8	−10	(−27)	−11	−13	-9_{8}	11
Potsdam	346·9	52·4 „	−2	−6	−6	−6	5	5	−1	−2	0	10	7	−1	8	−4	3	1_{15}	4
Rude Skov	347·5	55·8 „	...	...	...	...	8	......	5	−12	(−1?)	9	(15?)	10	...	−5	...	2_{6}	8
Munich	348·4	48·1 „	−4	14	0	−10	(−21)	?	?	4	18	−6	9	3	?	3	−17	1_{11}	8
Wilhelmshaven	351·9	53·5 „	−3	18	8	(−26)	−8	−18	(−10?)	9	−8	−2	−5	−7	?	2	?	-1_{11}	8
De Bilt	354·8	52·1 „	−3	−5	...	...	...	−7	4	−5	5	−5	−9	−6	3	−2	3	-5_{12}	5
Uccle	355·6	50·8 „	3	14	(23)	−14	−3	(36)	−7	−2	6	−17	7	0	−4	4	7	0_{13}	7
Val Joyeux	358·0	48·8 „	−7	−12	−6	−16	3	(−46)	(−34)	16	−7	−7	−8	14	16	11	4	0_{13}	10
Ebro	359·5	40·8 „	...	...	...	(39)	−2	(−28)	2	(−24)	−10	−11	...	9	...	−11	5	-3_{7}	7
Mean numerical residual			4_{25}	8_{23}	6_{20}	7_{20}	6_{26}	6_{21}	6_{18}	6_{27}	7_{20}	8_{23}	6_{24}	8_{26}	8_{15}	6_{27}	7_{27}		
The unit in this part of the Table is 1 second (not 0·1 minute, as above).	Mean P.M. residual		-7_{17}	4_{18}	3_{11}	-4_{13}	2_{18}	5_{8}	6_{14}	0_{4}	-24_{6}	-10_{16}	18_{9}	-41_{6}	1_{5}	-13_{10}	-10_{13}		
	Mean A.M. residual		6_{8}	-10_{5}	-5_{9}	-8_{7}	-8_{8}	1_{13}	-3_{4}	4_{18}	6_{14}	-2_{7}	-13_{15}	16_{20}	1_{10}	6_{17}	9_{14}		
	P.M.–A.M.		13	−14	−8	−4	−10	−4	−9	4	30	8	−31	57	0	19	19		
	"Day" residuals		-10_{6}	24_{14}	13_{5}	-27_{11}	1_{16}	8_{14}	8_{8}	9_{7}	-8_{13}	25_{6}	0_{17}	12_{7}	-18_{5}	-9_{20}	-2_{21}		
	"Night" residuals		-1_{19}	-35_{9}	-5_{15}	21_{9}	-4_{10}	-9_{7}	-5_{10}	1_{20}	7_{7}	-19_{17}	-4_{7}	0_{19}	10_{10}	23_{7}	7_{6}		
	"Night"–"Day"		9	−59	−18	48	−5	−17	−13	−8	15	−44	−4	−12	28	32	9		

* These observatories gave times in whole minutes only. For convenience the decimal 0 has been supposed added.

† These times were in error by positive amounts varying from 6 to 12 minutes.

‡ A small movement, of perhaps local character, nearly coincided with and obscured the precise commencement in this case.

§ Several observatories gave data for two movements in this case, the earliest one being at 8h. 30m. approximately. Only the later one is here considered, which necessitates exclusion of data from observatories which measured only the former one.

To face p. 212.

electromagnetic waves to other parts. He was not aware of any direct evidence confirmatory of Dr. Chapman's statement that the atmospheric currents causing the diurnal variation were mainly situated in the comparatively near neighbourhood of the station. Also, even if this were the case, he did not see that any inference could be drawn as to what happened in the case of so different a phenomenon as a "sudden commencement." We did not even know whether the electrical currents causing the two phenomena were at the same level in the atmosphere. "Sudden commencements" were sometimes large, 50γ or even 100γ. They were not instantaneous changes, the rise normally seen in horizontal force in ordinary latitudes taking usually four or five minutes to attain its full value, a curious feature being that the high value was generally retained for some time, sometimes, in fact, for several hours, especially at stations in certain latitudes. A consideration of the exact nature of the phenomenon was a desirable prelude to any theorising. On the historical question which Dr. Chapman had raised, as to the discovery of a tendency to "repetition" in magnetic storms, it was very natural for him as a Greenwich man to emphasise the really valuable work which Mr. Maunder had done in this connection, but it was only fair to remember the much earlier work of Broun, and the independent work of A. Harvey and the late Prof. Birkeland. One reason why a belief in the "repetition" of magnetic storms had advanced so slowly was that sunspot theories had been so often suggested by cranks that they were naturally somewhat suspect. In the case of magnetic disturbances, and Mr. Maunder's Greenwich lists were no exception to the rule, the great majority did not have a "sudden commencement," and opinions as to when the average storm began or ended might differ by several hours. It was thus in general impossible to assign an exact value to the interval between two storms. Further, what one man called a "storm" another did not, so where one might see a "repetition" another would not. A further complication was the existence of a diurnal period in disturbance data, representing partly statistical imperfections and partly a real natural phenomenon. Dr. Chapman did not seem to realise that the 27-day period was just as manifest in quiet as in disturbed conditions. It was also important to remember that it had presented itself in years like 1913, when sunspots were almost non existent, just as decisively as in years of many sunspots.

Prof. Newall said that when Maunder's results were first brought forward it was felt to be a difficulty that they should involve an almost geometrical recurrence; because, of course, the disturbances on the surface of the sun were travelling with varying periods. When the slide showing the periodic recurrence of certain storms was first shown by Maunder it was noticed that a 1 per cent. difference in the adopted rotation period would change the direction of the lines which indicated coincidence of period to 45 deg. from the vertical. One saw on looking at the slide that there were many cases in which lines at different angles would suit. No one familiar with eclipse phenomena could doubt, however, that something originated in the sun. Many of the streamers sent out from the corona seemed quite straight, and it was difficult to resist the idea that sometimes the earth would be at the other end. Had Dr. Chapman considered the difficulty raised by Kelvin of the necessary energy relations? The energy of the magnetic storm itself might be attributed to something abstracted from the energy of rotation of the earth by the particles after entering the atmosphere.

Father Cortie thought Maunder was the first to connect the 27-day period with the solar streamers. There seemed to him to be a difficulty about these streamers of charged particles. Schuster showed that owing to the mutual repulsion of the charges the stream would be dispersed. He did not think that any particular storm was associated with a particular sunspot, but with a disturbed area of the sun which might extend many degrees. His idea was that the dispersed streamers from the disturbed regions formed clouds of particles and that the earth may enter such a cloud, giving rise to magnetic storms. From the table it seemed to him that there was a slight preponderance of effect on the night side relative to Greenwich—*i.e.*, over the Pacific ocean. Would the presence of a large expanse of water be likely to affect the phenomena ?

Mr. T. **Smith** asked why the times of sudden commencement were chosen. He would imagine that similar time relationships would hold for any prominent feature. One would normally expect greater irregularities just at the beginning of the storm than when it was well under way. When it was known that a storm had commenced, wide scale runs could be started and measurements made on some subsequent outstanding feature.

A **Member** asked what irregularities might be due to lag in the different instruments used in the different observations.

Prof. **Lees** said that when Dr. Chree first brought the matter before the Society he had set some of his students to work out the correlation factor of Bauer's figures. The factor was so low that there was evidently no basis whatever for Bauer's theory.

Dr. **Chapman**, in reply, said he did not put any reliance on the actual figures given in the Paper. The observational error was too great. It was the method of treating the observations that he thought was of importance. He thought diurnal variations were produced at lower levels of the atmosphere than storms. By " near neighbourhood " he meant within about 1,000 kilometres or so. He did not think Maunder connected the storms with actual sunspots, but rather with disturbed regions. Of course, any proper motion of the disturbed region on the sun would alter the period in particular cases ; but in examining a long series it was best to take the synodic period since the others would be distributed on either side of this. He had not yet gone into the energy question, but did not think that this would present any insuperable difficulties. He did not think the streams would diffuse very much. He thought they consisted of α particles. They did not necessarily proceed radially from the sun, but might emerge in all directions from radial to tangential. This, however, would not seriously affect the time taken to pass across the earth. There was a definite reason for chosing the commencement times of the storms. When the storm is under way the magnetic state of the earth is fluctuating, and it is difficult to recognise accurately particular features. There was certainly a lag in the measurements which differed in different instruments ; but the effect of this was greatly reduced in the present method of grouping the observations, since the same instrument on different occasions contributes results to different groups. In this respect the method has a great advantage over classification on a geographical basis, in which case any instrument is always in the same group.

XVII. *The Entropy of a Metal.* By H. STANLEY ALLEN, M.A., D.Sc., *University of London, King's College.*

RECEIVED APRIL 12TH, 1918.

§ 1. According to the theory of Ratnowsky* the entropy of a gram atom of a substance in the solid state may be expressed in the form,

$$S=\frac{3Nk}{x^3}\left\{4\int_0^x\frac{\xi^3 d\xi}{e^\xi-1}-x^3\log\left(1-e^{-x}\right)\right\},$$

where N is Avogadro's constant, k is the gas constant for a single molecule, and $x=h\nu_m/kT=\beta\nu/T=\Theta/T$, h being Planck's constant, ν_m the maximum vibration frequency of Debye's theory, and Θ a temperature characteristic of the substance considered.

In a previous communication † I have shown that the correct form of the approximation for small values of x, corresponding to high values of the absolute temperature T, is

$$S=3Nk\left\{\frac{4}{3}-\log x+\frac{x^2}{40}\ldots\right\},$$

where $3Nk=C_\infty=5{\cdot}96$ calories.

§ 2. Up to the present time no direct evidence as to the validity of this equation has been brought forward, but an important Paper by Lewis and Gibson ‡ on the entropy of the elements now provides the data required for testing the formula. The work of these authors is based on the heat theorem of Nernst—the so-called third law of thermodynamics—which may be stated in the form: "In an isothermal process involving pure solids and liquids the change in entropy approaches zero as the temperature approaches the absolute zero." Following Planck, the principle may be re-stated in the more general form: "The entropy of every actual substance in the pure state is zero at the absolute zero of temperature." Hence

$$S=\int_0^T\frac{dQ}{T}=\int_0^T\frac{CdT}{T},$$

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* Ratnowsky, "Deutsch. Physikal. Gesell. Verh." Vol. XVI., p. 1033' 1914.

† H. S. Allen, "Proc. Phys. Soc." Vol. XXVIII., p. 302, 1916.

‡ Lewis and Gibson, "Am. Chem. Soc. Journ." Vol. XXXIX., p. 2554, 1917.

where C is the atomic heat. The absolute value of the entropy may be calculated, either at constant volume or at constant pressure, for any assigned temperature, provided C is known as a function of the temperature. The primary object of Lewis and Gibson was to determine the entropies of the elements in their standard states, at the standard temperature of free-energy measurements, namely 25°C. or 298°K. For many substances the atomic heat at constant volume is given by the equation

$$C_v = f(T/\theta),$$

where f is the same function for different substances, and θ a characteristic constant for each substance. Instead of making any assumption as to the exact form of the heat-capacity equation, the authors employed a graphic method of calculating the entropy of solid substances for which the equation has the same form. It is to be noted that θ is defined by them as the temperature at which C_v is one-half of the Dulong and Petit constant, so that when $T=\theta$, $C_v = 3R/2 = \frac{1}{2}\, C_\infty$.

§ 3. The following table contains the values of the characteristic constants and the entropies for seven metals which have been made the subject of accurate investigations at low temperatures. The column headed θ gives the characteristic temperature as defined by Lewis and Gibson, Θ is the characteristic temperature employed by Debye.* The entropy in calories per degree at 298° K., under the condition of constant volume, is given in the last two columns of the table. The observed values are quoted from the Paper by Lewis and Gibson,† whilst the calculated values have been obtained from the approximate form of the formula of Ratnowsky. The agreement between the observed and the calculated

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* It is noteworthy that the values of Θ in the table are almost exactly four times the corresponding values of θ. It is interesting to test this relationship assuming Debye's expression for the specific heat, which may be written

$$\frac{C_v}{C_\infty} = \frac{12}{x^3}\int_0^x \frac{\xi^3 d\xi}{e^{\xi}-1} - \frac{3x}{e^{x}-1}.$$

I have to thank Prof. J. B. Dale for calculating the value of the right hand side of this equation when $x=4$, that is when $\Theta = 4T$. It is found that $C_v/C_\infty = 0{\cdot}503059$, which differs from one-half by less than one per cent. This result shows that when Debye's theory is employed, the characteristic temperature, Θ, is almost, but not exactly, equal to four times the temperature at which C_v is one-half of the limiting value C_∞.

† These authors also give values for the entropy of the elements under constant pressure.

values is remarkably good. The letter *a* after an observed value indicates that the value in question is hardly likely to be in error by more than one-third of an entropy unit.

ENTROPY OF THE METALS AT 298° K.

Element.	θ.	Θ.	S_c, observed.	S_c, calculated.
Aluminium	95·5	382 S	6·73 *a*	6·72
Iron	99·1	385 G	6·54	6·68
Copper	78·2	315 K	7·91 *a*	7·79
Zinc	57·6	230 G, N	9·60 *a*	9·59
Silver	53·7	215 G	10·00 *a*	9·98
Cadmium	42·4	168 G	11·38	11·42
Lead	22·0	88 K, S	15·11 *a*	14·89

G=Griffiths, K=Keesom and Onnes, N=Nernst, S=Schwers.

The thermal behaviour of sodium [*] is somewhat exceptional, and for that reason it has not been included in the table. At low temperatures Messrs. Griffiths find $\Theta=180$, which yields as the calculated value of the entropy 12·21 calories per degree, whilst the value deduced by Lewis and Gibson from Dewar's measurements is 11·43.

Another case in which it is possible to institute a comparison between the theoretical and the experimental value for the entropy is that of solid mercury at the temperature of the melting point. Nernst and Lindemann found $\Theta=96{\cdot}6$, so that at the melting point ($T=234{\cdot}1°$ K.) the entropy is 13·26 calories per degree. The corresponding value deduced by Lewis and Gibson from the observations on the specific heat is 13·31 calories per degree. Here the agreement could scarcely be improved upon.

§ 4. When x is not too large, the value of the entropy can be calculated from the formula of Ratnowsky expanded in the form

$$S=C_{\infty}\left\{\frac{4}{3}-\log x+\frac{x^2}{40}-\frac{x^4}{2240}+\frac{x^6}{108864}-\ldots\right\}.$$

The series is convergent when x is (numeraiclly) less than 2π, but converges slowly when x is as large as 4. For large values of x the method of expansion given by Debye [†] may be employed, or by combining the expression of Ratnowsky for the entropy with that of Debye for the specific heat the entropy may be found from the formula,

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[*] E. H. Griffiths and E. Griffiths, "Phil. Trans," A. Vol. CCXIV., p. 319, 1914. Eastman and Rodebush, "Journ. Am. Chem. Soc." Vol. XL., p. 489, 1918.

[†] Debye, "Annalen der Physik." Vol. XXXIX., p. 789, 1912.

$$S=\tfrac{1}{3}C_v+C^{\infty}\left[\frac{x}{e^x-1}-\log\left(1-e^{-x}\right)\right],$$

in which the values for C_v calculated by Debye may be inserted.

In the diagram (Fig. 1) the value of the entropy in calories per degree, calculated as described above, has been plotted against the value of $1/x=T/\Theta$. For any assigned value of Θ this gives an entropy temperature diagram. In the same figure are given the points (marked ×) determined from the values of S_v recorded by Lewis and Gibson as corresponding to selected values of θ/T. In plotting these points it has been assumed that $\Theta=4{\cdot}021\theta$, which is the relation given by Debye as corresponding to $C_v=\tfrac{1}{2}C_\infty$. It will be seen that the two sets of points lie on a smooth curve, showing the close agreement between the theoretical and the experimental results throughout the whole range of temperatures considered.

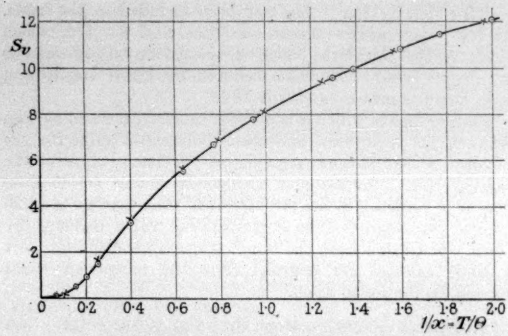

FIG. 1.—VARIATION OF ENTROPY WITH TEMPERATURE.
⊙ Formula of Ratnowsky. × Lewis and Gibson.

§ 5. The above results prove that the formula of Ratnowsky gives values for the entropy of a solid in very close agreement with those obtained directly from observations on the specific heat at various temperatures. It therefore becomes a matter of special interest to examine the principal assumptions made in the deduction of the formula. These appear to be four in number.

1. The whole internal energy, U, of the gram atom is made up of two parts, the "vibrational energy," E, and the potential energy $f(v)$, which is a function of the volume. Even at the absolute zero of temperature the substance possesses a certain amount of potential energy.

2. Boltzmann's equation for the entropy is assumed, so that

$$S=k \log W+C,$$

where W is the number of "complexions" of the system when it has a given energy. The constant of integration, C, is ignored or included as a constant multiplier in the quantity W, so that the equation reduces to

$$S=k \log W.$$

This, as Planck points out ("Heat Radiation," § 120), is equivalent to assigning a definite absolute value to the entropy S, and leads necessarily to Nernst's heat theorem.

3. It is assumed that

$$\frac{\partial S}{\partial E}=\frac{1}{T}.$$

This implies a differentiation at constant volume.

4. Following Debye, it is assumed that the number of vibrations within a frequency interval $d\nu$ is given by

$$9N\nu^2 d\nu/\nu^3{}_m,$$

where ν_m denotes the maximum vibration frequency. It is known that Debye's theory must be regarded as giving only a first approximation, since the existence of a sharply defined maximum frequency is a somewhat arbitrary assumption. Nevertheless the formula deduced for the atomic heat gives a very close approximation to the experimental results.

§ 6. We may conclude that the hypotheses assumed by Ratnowsky in dealing with the entropy of a solid are probably justified as being at least approximately true. It is necessary to point out, however, that the results of the present inquiry have no bearing on the further assumption made by Ratnowsky that his expression for the entropy may be applied to the liquid state. This is a question which I have discussed in a former Paper dealing with the latent heat of fusion of a metal (*loc. cit.*), but it cannot be definitely decided till further information is available as to the applicability of Debye's theory to liquids.

ABSTRACT.

An expression for the entropy of one gram atom of a substance in the solid state has been given by Ratnowsky. In a communication to the Physical Society in 1916 the author gave the correct form of the approximation required for high values of the absolute temperature in terms of Bernoulli's numbers. The data required for testing the formula have been supplied in a recent Paper by Lewis and Gibson, who have given values for the entropy of the elements under the condition of constant volume, and also under constant pressure. These values were deduced from observations on the specific heat assuming the truth of the heat theorem of Nernst, that the entropy of every actual substance in the pure state is zero at the absolute zero of temperature. It is found that the formula of Ratnowsky gives values for the entropy of a solid in very close agreement with those obtained by Lewis and Gibson. The hypotheses assumed in the theory of Ratnoswky are discussed, and the conclusion is drawn that these are probably justified as being at least approximately true.

XVIII. *On Tracing Rays Through an Optical System.* (*Second Paper.*) *By* T. Smith, *B.A.* (*From the National Physical Laboratory.*)

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In the earlier communication[*] bearing this title the author gave formulæ by which a ray in three dimensions could be traced through a system of coaxial spherical refracting surfaces. These formulæ are algebraic, and do not require the use of any tables; in this respect they are in marked contrast to those generally used. The new formulæ are primarily intended for use in conjunction with a calculating machine; the older trigonometrical formulæ have been arranged for logarithmic work. Experience has shown that logarithmic calculations take several times as long as the corresponding mechanical operations.

The present Paper contains the modification of these formulæ generally used for rays in an axial plane, and, for both two and three dimensional cases, new formulæ which are universally applicable, so that the calculations could, if required, be carried out entirely by mechanical means. In the three dimensional case a modification is only required for the preliminary calculation described in the previous Paper. The present Paper also includes the method followed in calculations relating to transverse focal lines.

By far the greater number of rays that have to be traced through systems of lenses lie in a plane containing the axis. The system used in the previous Paper can be modified for these cases. Let φ_λ, φ'_λ be the angles of incidence and emergence at the surface which separates media of refractive indices $\mu_{\lambda-1}$ and μ_λ. The inclination to the axis of the ray between surfaces λ and $\lambda+1$ is ψ_λ; t_λ and a_λ[†] are the distances between the vertices and the centres of curvature respectively of these surfaces. The radius of curvature of the surface λ is r_λ, and its curvature is $R_\lambda=1/r_\lambda$. The length of the perpendicular from the centre of curvature of this surface on to the incident ray is h_λ[†], and on to the refracted ray is h'_λ. To find the position of the final emergent ray the formulæ

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[*] Proc. Phys. Soc., Vol. XXVII., page 502.

[†] There are minor alterations from the notation employed in the previous Paper.

$$
\begin{aligned}
h_\lambda &= h'_{\lambda-1}+a_{\lambda-1}\sin\psi_{\lambda-1} & (1)\\
\mu_\lambda h'_\lambda &= \mu_{\lambda-1}h_{\lambda-1} & (2)\\
\sin\varphi_\lambda &= h_\lambda R_\lambda & (3)\\
\sin\varphi'_\lambda &= h'_\lambda R_\lambda & (4)\\
\psi_\lambda &= \varphi'_\lambda-\varphi_\lambda+\psi_{\lambda+1} & (5)\\
a_\lambda &= t_\lambda-r_\lambda+r_{\lambda-1} & (6)
\end{aligned}
$$

are successively applied at each refracting surface.

Of these formulæ, (1), (2), (3), (4), (6) are equivalent to some of the general formulæ contained in the previous Paper. Equation (5) constitutes a modification which applies only to the particular case of rays in two dimensions. The formulæ as given above involve reference to tables of natural sines, since equation (5) refers to angles, and not to their trigonometrical functions. On paper, owing to the algebraic simplicity of equation (5), the system appears to be less complex than before; in fact, the simplification is illusory, for it actually involves at least three references to tables for each surface, and these take up most of the time and also involve most chance of error. The system as given above is the form that is generally used; but experience has shown that it is quite as quick to avoid reference to tables at all. The calculations are generally arranged in three parallel columns as follows:—

$$
\begin{tikzcd}
\text{Angle.} & \text{Sine.} & \text{Central perpendicular.} \arrow[d] \\
\psi_0 \arrow[r] & \sin\psi_0 & {} \arrow[d] \\
\varphi_1 & \sin\varphi_1 \arrow[l] & h_1 \arrow[l] \arrow[d] \\
\varphi_1' \arrow[d] & \sin\varphi_1' \arrow[l] & h_1' \arrow[l] \\
\psi_1 \arrow[r] & \sin\psi_1 \arrow[r] & h_2-h_1' \arrow[d] \\
\varphi_2 & \sin\varphi_2 \arrow[l] & h_2 \arrow[l] \arrow[d] \\
\varphi_2' \arrow[d] & \sin\varphi_2' \arrow[l] & h_2' \arrow[l] \\
\psi_2 \arrow[r] & \sin\psi_2 \arrow[r] & h_3-h_2' \arrow[d] \\
 & & {}
\end{tikzcd}
$$

The arrows show the order in which the operations are carried out.

It frequently happens that, after the ray has been traced, it is required to find the positions at which it is intersected by the focal lines formed by a pencil of neighbouring rays. The calculation of the radial focal line was considered in the previous Paper. It is most simply effected by finding the length of the ray intercepted between successive surfaces and the power of the various surfaces for that ray. The formula for the intercepted length d_λ corresponding to the axial length t_λ is

$$d_\lambda = a_\lambda \cos \psi_\lambda + r_\lambda \cos \varphi'_\lambda - r_{\lambda+1} \cos \varphi_{\lambda+1} \quad . \quad . \quad (7)$$

and the power of the surface for this ray, K_λ, is given by

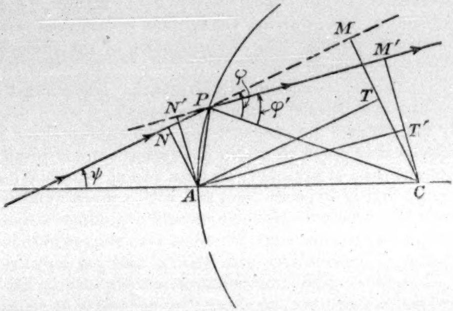
$$K_\lambda = (\mu_\lambda \cos \varphi'_\lambda - \mu_{\lambda-1} \cos \varphi_\lambda) R_\lambda \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The position of the radial focal line is found by regarding d_λ and K_λ as t_λ and κ_λ, and applying the formulæ for paraxial rays to find conjugate points. The determination of the transverse focal line is considered later.

With the above treatment, as with all other systems that the author has met, special formulæ are necessary when flat or nearly flat surfaces are encountered. The application of the formulæ as they stand to nearly flat surfaces would result in considerable loss of accuracy, because two successive a's are very great, but of opposite sign, and with a given number of figures in $\sin \psi$ the sum of the two successive products $a \sin \psi$ is uncertain to an extent which increases with the magnitude of the radius. To avoid this uncertainty, and yet be able to treat all surfaces alike, whether they are flat, nearly flat or very considerably curved, an alternative method of calculation has been devised.

Consider for a moment the conditions which such a system must satisfy. In the first place, all points of reference must be at a finite distance from the portions of the surfaces operative in producing refraction; thus, no reference is possible to the centre of curvature or to the point of intersection of a ray with the axis, since either may be at infinity. Again, the radius of curvature may not be used, for this may become infinite; on the other hand, its reciprocal, the curvature, may be employed, since it is always finite or zero. More generally, no lengths measured along the axis may be used if high accuracy is desired, because these are so variable in magnitude. Transverse distances, on the other hand, vary within small limits fixed by the apertures of the various lenses, and their use will tend to give uniform reliability at all surfaces. Lastly, if the formulæ involve fractions, the denominators must in all cases be essentially constant in sign and large in magnitude. It will be shown that all these conditions can be satisfied.

Take, first, a ray which lies in an axial plane. The figure shows the incident ray NP refracted at P as the emergent ray PM'. Let C be the centre of curvature of the refracting surface, and A the point at which the axis of the system meets the surface. Let M, M' be the feet of the perpendiculars to the incident and refracted rays from the centre of curvature C, and N, N' the feet of those from the vertex A. The diagram shows the case in which the various quantities entering the formulæ are positive. Light travels from left to right, and the radius of curvature is positive for a surface presenting its convex side to the incident light. The inclinations ψ, ψ' of the rays to the axis are positive when the rotation of a straight line from the direction of the axis through an acute angle in the counter-clockwise direction will bring it into parallelism with the ray. Similarly, the angles φ, φ' of incidence and emergence are positive when a rotation through an acute angle in the counterclockwise direction from the direction of the normal to the surface at the point of refraction brings the line into coincidence with the direction of the ray. AT and AT' are perpendiculars from A to CM and CM'. The radius and the central perpendiculars will be denoted as before by r,



h and h', and the perpendiculars AN, AN' by $\mathfrak{h}$ and $\mathfrak{h}'$. The chord AP may be denoted by $\mathfrak{H}$. From the figure

$$CM = r \sin \varphi,$$
$$CT = r \sin \psi,$$
$$AN = CM - CT,$$

or

$$\mathfrak{h} = r \sin \varphi - r \sin \psi.$$

In this form the equation may not be used, because it involves r. Dividing throughout by r and rearranging, the equation becomes

$$\sin \varphi = \sin \psi + \mathfrak{h} R \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

a form which is free from objection. If, then, the incident ray is given by ψ and $\mathfrak{h}$, equation (9) enables the angle of incidence to be found. The angle of emergence is given by the law of refraction

$$\mu' \sin \varphi' = \mu \sin \varphi$$

and the value of ψ' by

$$\psi' = \varphi' - \varphi + \psi.$$

At the next surface the length of the perpendicular from the vertex to the incident ray is evidently

$$\mathfrak{h}' + t \sin \psi',$$

where t is the axial distance between the two vertices. It therefore only remains to determine the value of $\mathfrak{h}'$. This may not be found from the equation,

$$\mathfrak{h}' = AN' = CM' - CT' = r(\sin \varphi' - \sin \psi'),$$

since this involves r. Neither may r be eliminated by (9), because the denominator in

$$\mathfrak{h}' = \mathfrak{h} \frac{\sin \varphi' - \sin \psi'}{\sin \varphi \ - \sin \psi}$$

becomes very small for long radii. Since, however, $\varphi' - \psi' = \varphi - \psi$, this equation may be written

$$\frac{\mathfrak{h}'}{\mathfrak{h}} = \frac{\cos \frac{1}{2}(\varphi' + \psi')}{\cos \frac{1}{2}(\varphi + \psi)} = \frac{\cos \varphi' + \cos \psi'}{\cos \varphi + \cos \psi},$$

and in this form the conditions laid down above are satisfied.* As the cosines of φ, φ', ψ and ψ' are required in dealing with the refraction of neighbouring rays, these are the quantities that will normally be used. The equations of the new system are thus

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* It may be noted that $\mathfrak{h}/\cos \frac{1}{2}(\varphi + \psi) = \mathfrak{h}'/\cos \frac{1}{2}(\varphi' + \psi') = \mathfrak{H}$, since the angles PAN, PAN' are equal to $\frac{1}{2}(\varphi + \psi)$ and $\frac{1}{2}(\varphi' + \psi')$ respectively.

$$\sin \varphi_\lambda = \sin \psi_{\lambda-1} + \mathfrak{h}_\lambda R_\lambda . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\mu_\lambda \sin \varphi'_\lambda = \mu_{\lambda-1} \sin \varphi_\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\psi_\lambda = \varphi'_\lambda - \varphi_\lambda + \psi_{\lambda-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\mathfrak{h}'_\lambda = \mathfrak{h}_\lambda \frac{\cos \varphi'_\lambda + \cos \psi_\lambda}{\cos \varphi_\lambda + \cos \psi_{\lambda-1}} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\mathfrak{h}_{\lambda+1} = \mathfrak{h}'_\lambda + t_\lambda \sin \psi_\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The calculations are arranged very similarly to those set out diagrammatically above. Four columns are used—for the angles, their sines, their cosines and the perpendiculars from the vertices. It may be noted that the quantities found suffice to check all the calculations, since both the sines and the cosines of all the angles are known. In particular, absolute checks are available for all the quantities abstracted from the tables.

The power of the surface λ for the ray considered is derived as before from (8), but equation (7) has to be replaced. Several formulæ are available. If $\mathfrak{H}$ has been determined for the various surfaces, projection upon the axis gives

$$d_\lambda \cos \psi_\lambda = t_\lambda - \tfrac{1}{2} \mathfrak{H}^2_\lambda R_\lambda + \tfrac{1}{2} \mathfrak{H}^2_{\lambda+1} R_{\lambda+1} \quad . \quad . \quad . \quad . \quad (13)$$

or projection upon the ray leads to

$$d_\lambda = t_\lambda \cos \psi_\lambda - \sqrt{(\mathfrak{H}^2_\lambda - \mathfrak{h}'^2_\lambda)} + \sqrt{(\mathfrak{H}^2_{\lambda+1} - \mathfrak{h}^2_{\lambda+1})} \ . \quad (14)$$

When $\mathfrak{H}$ is unknown, perhaps the simplest formula which does not involve the determination of any new quantities is

$$d_\lambda = t_\lambda \cos \psi_\lambda - \mathfrak{h}'_\lambda \frac{\sin \varphi'_\lambda + \sin \psi_\lambda}{\cos \varphi'_\lambda + \cos \psi_\lambda} + \mathfrak{h}_{\lambda+1} \frac{\sin \varphi_{\lambda+1} + \sin \psi_\lambda}{\cos \varphi_{\lambda+1} + \cos \psi_\lambda} \quad (15)$$

Other interesting formulæ are obtained by projecting on to a line bisecting the angle between the ray and the axis. A simple formula of this class is

$$d_\lambda - t_\lambda = (\mathfrak{H}_{\lambda+1} \sin \tfrac{1}{2}\varphi_{\lambda+1} - \mathfrak{H}_\lambda \sin \tfrac{1}{2}\varphi'_\lambda) \sec \tfrac{1}{2}\psi_\lambda \, .$$

From this it follows that the optical length of the ray between the feet of perpendiculars to the incident and final emergent paths from the two external vertices exceeds the axial path between these vertices by

$$\tfrac{1}{2}\Sigma\mathfrak{H}_{\lambda}\mu_{\lambda}\sin\varphi'_{\lambda}\{\sec\tfrac{1}{2}\psi_{\lambda-1}\sec\tfrac{1}{2}\varphi_{\lambda}-\sec\tfrac{1}{2}\psi_{\lambda}\sec\tfrac{1}{2}\varphi'_{\lambda}\}$$

$$=\tfrac{1}{4}\Sigma\frac{\mathfrak{H}_{\lambda}\,\mu_{\lambda}\sin\varphi_{\lambda}'}{\cos\tfrac{1}{2}\psi_{\lambda-1}\cos\tfrac{1}{2}\psi_{\lambda}\cos\tfrac{1}{2}\varphi_{\lambda}\cos\tfrac{1}{2}\varphi'_{\lambda}}\{\cos\tfrac{1}{2}(\varphi'_{\lambda}+\psi_{\lambda})$$

$$-\cos\tfrac{1}{2}(\varphi_{\lambda}+\psi_{\lambda-1})\}$$

$$\doteq\tfrac{1}{4}\Sigma\frac{(\mathfrak{h}'_{\lambda}-\mathfrak{h}_{\lambda})\mu_{\lambda}\sin\varphi'_{\lambda}}{\cos\tfrac{1}{2}\psi_{\lambda-1}\cos\tfrac{1}{2}\psi_{\lambda}\cos\tfrac{1}{2}\varphi_{\lambda}\cos\tfrac{1}{2}\varphi'_{\lambda}}$$

$$=\Sigma\frac{(\mathfrak{h}'_{\lambda}-\mathfrak{h}_{\lambda})\mu_{\lambda}\sin\varphi'_{\lambda}}{\sqrt{\{(1+\cos\psi_{\lambda-1})(1+\cos\psi_{\lambda})(1+\cos\varphi_{\lambda})(1+\cos\varphi'_{\lambda})\}}}. \quad (16)$$

A number of other forms can be found for this path difference, but it is unnecessary to pursue the subject further here.

Before leaving the consideration of a ray lying in an axial plane, it may be noted that the two series of formulæ using the centres of curvature and the vertices of the surfaces respectively as reference points may be used in conjunction with one another. Reference to the centres of curvature would be made for short radii, and to the vertices for long radii. All the formulæ remain as before, except that, on changing from one system to the other, instead of a_{λ} or t_{λ}, the axial distance between the two reference points is used. For example, if the centre of curvature is the reference point for surface λ and the vertex for surface $\lambda+1$, equation (1) will be replaced by

$$\mathfrak{h}_{\lambda+1}=h'_{\lambda}+b_{\lambda}\sin\psi_{\lambda},$$

where $$b_{\lambda}=t_{\lambda}-r_{\lambda}.$$

If at surface $\lambda+2$ the centre of curvature is employed again, the transition equation will be

$$h_{\lambda+2}=\mathfrak{h}'_{\lambda+1}+c_{\lambda+1}\sin\psi_{\lambda+1},$$

where $$c_{\lambda+1}=t_{\lambda+1}+r_{\lambda+2}.$$

The corresponding changes in the formulæ for d are obvious, and need not be set down here.

The extension of the formulæ to rays which do not always lie in the same plane may now be considered. As in the earlier Paper, let the ray incident in the medium $\mu_{\lambda-1}$ on the surface λ be determined in direction by the direction cosines $L_{\lambda-1}$, $M_{\lambda-1}$, $N_{\lambda-1}$; and the refracted ray in the medium μ_{λ} by L_{λ}, M_{λ}, N_{λ}. The direction cosines of the normal to the surface at the point where refraction takes place are l_λ, m_λ, n_λ. The previous notation involving φ and ψ may be retained also, so that

$$L_\lambda = \cos\psi_\lambda$$

and

$$L_{\lambda-1}l_\lambda + M_{\lambda-1}m_\lambda + N_{\lambda-1}n_\lambda = \cos\varphi_\lambda$$

$$L_\lambda l_\lambda + M_\lambda m_\lambda + N_\lambda n_\lambda = \cos\varphi'_\lambda$$

In the two dimensional case the perpendicular from the vertex to the incident ray made angles $\frac{\pi}{2}+\psi_{\lambda-1}$ and $\frac{\pi}{2}+\varphi_\lambda$ with the axis and the normal to the surface. In the general case this will no longer be true. Let the direction cosines of the two perpendiculars from the vertex to the incident and emergent rays be

$$e_\lambda,\ f_\lambda,\ g_\lambda$$

and

$$e'_\lambda,\ f'_\lambda,\ g'_\lambda$$

respectively. The co-ordinates of the feet of the perpendiculars are thus

$$e_\lambda \mathfrak{h}_\lambda,\ f_\lambda \mathfrak{h}_\lambda,\ g_\lambda \mathfrak{h}_\lambda$$

and

$$e'_\lambda \mathfrak{h}'_\lambda,\ f'_\lambda \mathfrak{h}'_\lambda,\ g'_\lambda \mathfrak{h}'_\lambda$$

where the vertex of the λth surface is taken as origin. The point of refraction is

$$r_\lambda(1-l_\lambda),\ -r_\lambda m_\lambda,\ -r_\lambda n_\lambda,$$

and the lengths intercepted on the rays between the perpendiculars and this point are $\sqrt{(\mathfrak{H}^2_\lambda - \mathfrak{h}^2_\lambda)}$ and $\sqrt{(\mathfrak{H}^2_\lambda - \mathfrak{h}'^2_\lambda)}$. Therefore

$$\sqrt{(\mathfrak{H}^2_\lambda - \mathfrak{h}^2_\lambda)} = \frac{r_\lambda(1-l_\lambda) - e_\lambda \mathfrak{h}_\lambda}{L_{\lambda-1}} = -\frac{r_\lambda m_\lambda + f_\lambda \mathfrak{h}_\lambda}{M_{\lambda-1}} = -\frac{r_\lambda m_\lambda + g_\lambda \mathfrak{h}_\lambda}{N_{\lambda-1}} \tag{17}$$

Multiply the fractions, both numerator and denominator, by e_λ, f_λ, g_λ and add. Since

$$e_\lambda L_{\lambda-1} + f_\lambda M_{\lambda-1} + g_\lambda N_{\lambda-1} = 0,$$

it follows that

$$r_\lambda\{e_\lambda - (e_\lambda l_\lambda + f_\lambda m_\lambda + g_\lambda n_\lambda)\} = \mathfrak{h}_\lambda$$

or

$$e_\lambda l_\lambda + f_\lambda m_\lambda + g_\lambda n_\lambda = e_\lambda - \mathfrak{h}_\lambda R_\lambda \quad . \quad . \quad . \quad . \tag{18}$$

the equation which leads to the value of the angle of incidence in the two dimensional case. Again multiply the fractions by $1+l_\lambda$, m_λ, n_λ, and add as before; this gives

$$\sqrt{(\mathfrak{H}^2_\lambda - \mathfrak{h}^2_\lambda)} = -(e_\lambda + e_\lambda l_\lambda + f_\lambda m_\lambda + g_\lambda n_\lambda)\mathfrak{h}_\lambda / (\cos\psi_{\lambda-1} + \cos\varphi_\lambda)$$

$$= (\mathfrak{h}_\lambda{}^2 R_\lambda - 2e_\lambda \mathfrak{h}_\lambda)/(\cos\psi_{\lambda-1} + \cos\varphi_\lambda) \quad . \quad . \tag{19}$$

by (18). Also multiplying by $L_{\lambda-1}$, $M_{\lambda-1}$, $N_{\lambda-1}$ and adding

$$\sqrt{(\mathfrak{H}_\lambda - \mathfrak{h}^2_\lambda)} = r_\lambda(\cos\psi_{\lambda-1} - \cos\varphi_\lambda). \quad . \quad . \quad . \tag{20}$$

The elimination of $\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}_\lambda^2)}$ between (19) and (20) leads to

$$\cos^2\varphi_\lambda=\cos^2\psi_{\lambda-1}+2e_\lambda\mathfrak{h}_\lambda R_\lambda-\mathfrak{h}_\lambda^2R_\lambda^2 \quad . \quad . \quad . \quad . \quad (21)$$

as an equation for the determination of φ_λ. The value of φ_λ' will be derived from

$$\mu_\lambda\sin\varphi_\lambda'=\mu_{\lambda-1}\sin\varphi_\lambda$$

and K_λ from

$$K_\lambda=(\mu_\lambda\cos\varphi_\lambda'-\mu_{\lambda-1}\cos\varphi_\lambda)R_\lambda.$$

When φ_λ has been found $\mathfrak{H}_\lambda^2$ is derived from (19). Equation (20) cannot, of course, be used for this purpose. It is next necessary to determine ψ_λ. This could be derived directly from the law of refraction

$$\mu_\lambda\cos\varphi_\lambda'-\mu_{\lambda-1}\cos\varphi_\lambda=\frac{\mu_\lambda\cos\psi_\lambda-\mu_{\lambda-1}\cos\psi_{\lambda-1}}{l_\lambda}$$

$$=\frac{\mu_\lambda\cos\psi_\lambda-\mu_{\lambda-1}\cos\psi_{\lambda-1}}{1-\frac{1}{2}\mathfrak{H}_\lambda^2R_\lambda^2}$$

but it is preferable to find $\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}_\lambda'^2)}$ directly, and thence obtain ψ_λ, using a relation which may be written down by analogy with (20)—viz.,

$$\cos\psi_\lambda=\cos\varphi_\lambda'+R_\lambda\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}_\lambda'^2)}. \quad . \quad . \quad . \quad . \quad (22)$$

The reason for this procedure is of course that, under the conditions which have been laid down, $\cos\psi_\lambda$ can be found from the value of $\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}_\lambda'^2)}$; but the process may not be reversed. To obtain an expression for $\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}_\lambda'^2)}$, write the equation of refraction in the form

$$\mu_\lambda r_\lambda(\cos\psi_\lambda-\cos\varphi_\lambda')=\mu_{\lambda-1}r_\lambda(\cos\psi_{\lambda-1}-\cos\varphi_\lambda)$$
$$-(1-l_\lambda)r_\lambda(\mu_\lambda\cos\varphi_\lambda'-\mu_{\lambda-1}\cos\varphi_\lambda)$$

or $$\mu_\lambda\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}_\lambda'^2)}=\mu_{\lambda-1}\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}_\lambda^2)}-\tfrac{1}{2}\mathfrak{H}_\lambda^2K_\lambda. \quad . \quad . \quad (23)$$

The equation

$$e_\lambda'\mathfrak{h}_\lambda'=\tfrac{1}{2}\mathfrak{H}_\lambda^2R_\lambda-\cos\psi_\lambda\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}_\lambda'^2)} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

derived from a relation similar to (17), completes the determination of the three quantities $\mathfrak{h}_\lambda'^2$, $e_\lambda'\mathfrak{h}_\lambda'$ and ψ_λ from the given values of $\mathfrak{h}_\lambda^2$, $e_\lambda\mathfrak{h}_\lambda$ and $\psi_{\lambda-1}$.

The length on the ray in medium μ_λ between the perpendiculars from the vertices of the surfaces λ and $\lambda+1$ is evidently $t_\lambda \cos\psi_\lambda$, and therefore

$$t_\lambda \cos\psi_\lambda = \frac{e_{\lambda+1}\mathfrak{h}_{\lambda+1} - e'_\lambda \mathfrak{h}'_\lambda + t_\lambda}{L_\lambda} = \frac{f_{\lambda+1}\mathfrak{h}_{\lambda+1} - f'_\lambda \mathfrak{h}_\lambda}{M_\lambda}$$

$$= \frac{g_{\lambda+1}\mathfrak{h}_{\lambda+1} - g'_\lambda \mathfrak{h}'_\lambda}{N_\lambda},$$

or $$e_{\lambda+1}\mathfrak{h}_{\lambda+1} = e'_\lambda \mathfrak{h}'_\lambda - t_\lambda \sin^2\psi_\lambda, \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and $$\mathfrak{h}^2_{\lambda+1} = \mathfrak{h}'^2_\lambda - 2e'_\lambda \mathfrak{h}'_\lambda t_\lambda + t^2_\lambda \sin^2\psi_\lambda \quad . \quad . \quad (26)$$

are the equations of transference to the new surface.

To complete the calculations on the lines laid down in the previous Paper, formulæ are required for d_λ. Either of the formulæ given earlier,

$$d_\lambda \cos\psi_\lambda = t_\lambda - \tfrac{1}{2}\mathfrak{H}^2_\lambda R_\lambda + \tfrac{1}{2}\mathfrak{H}^2_{\lambda+1} R_{\lambda+1}, \quad . \quad . \quad . \quad . \quad (13)$$

or $$d_\lambda = t_\lambda \cos\psi_\lambda - \sqrt{(\mathfrak{H}^2_\lambda - \mathfrak{h}'^2_\lambda)} + \sqrt{(\mathfrak{H}^2_{\lambda+1} - \mathfrak{h}^2_{\lambda+1})}, \quad (14)$$

may be used for this purpose. The latter is usually more convenient.

It should be noted that in the preliminary calculations to which the above equations relate, neither the sines of the angles involved nor the angles themselves need be found. The equations relate entirely to cosines with the exception of

$$\mu_\lambda \sin\phi'_\lambda = \mu_{\lambda-1} \sin\phi_\lambda,$$

and this equation is in practice squared and written as a cosine equation,

$$\cos^2\phi'_\lambda = 1 - \frac{\mu^2_{\lambda-1}}{\mu^2_\lambda}(1 - \cos^2\phi_\lambda).$$

The use of cosines instead of sines would be objectionable in a system in which reference to tables is involved, since a knowledge of the cosine of small angles does not determine the angle itself to high accuracy. This does not apply to the system outlined above, where the object is not to know the final angles of emergence, but to determine the K's, d's and the cosines of the various angles of incidence and refraction. The position of the emergent ray is obtained from a secondary calculation, involving these quantities, as shown in the earlier Paper.

As in the two dimensional case, the formulæ just obtained may be combined with those contained in the previous Paper. The equations of transference are easily found and need no discussion. It will be noted that many alternative forms can be given to the equations for performing the calculations consistently with the conditions laid down earlier. The system adopted is to be regarded as an illustration of a possible arrangement, and is not necessarily the best of its class.

The operations may be summarised as follows: Given $\mathfrak{h}_\lambda^2$, $e_\lambda \mathfrak{h}_\lambda$, $\cos\psi_{\lambda-1}$, it is required to find $\mathfrak{h}^2_{\lambda+1}$, $e_{\lambda+1}\mathfrak{h}_{\lambda+1}$, $\cos\psi_\lambda$ and also d_λ, K_λ, $\cos\varphi_\lambda$, $\cos\varphi'_\lambda$.

$$\cos^2\varphi_\lambda=\cos^2\psi_{\lambda-1}-(\mathfrak{h}_\lambda^2 R_\lambda-2e_\lambda\mathfrak{h}_\lambda)R_\lambda \qquad \text{gives } \cos\varphi_\lambda \quad (21)$$

$$\mu_\lambda^2(1-\cos^2\varphi'_\lambda)=\mu^2_{\lambda-1}(1-\cos^2\varphi_\lambda) \qquad \text{,,} \quad \cos\varphi' \quad (10)$$

$$K_\lambda=(\mu_\lambda\cos\varphi'_\lambda-\varphi_{\lambda-1}\cos\varphi_\lambda)R_\lambda \qquad \text{,,} \quad K_\lambda \quad (8)$$

$$\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}_\lambda^2)}=\frac{\mathfrak{h}_\lambda^2 R_\lambda-2e_\lambda\mathfrak{h}_\lambda}{\cos\psi_{\lambda-1}+\cos\varphi_\lambda} \qquad \text{,,} \quad \mathfrak{H}_\lambda^2 \quad (19)$$

$$\mu_\lambda\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}'^2_\lambda)}=\mu_{\lambda-1}\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}_\lambda^2)}-\tfrac{1}{2}\mathfrak{H}_\lambda^2 K_\lambda \qquad \text{,,} \quad \mathfrak{h}'^2_\lambda \quad (23)$$

$$\cos\psi_\lambda=\cos\varphi'_\lambda+R_\lambda\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}'^2_\lambda)} \qquad \text{,,} \quad \cos\psi_\lambda \quad (22)$$

$$e'_\lambda\mathfrak{h}'_\lambda=\tfrac{1}{2}\mathfrak{H}_\lambda^2 R_\lambda-\cos\psi_\lambda\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}'^2_\lambda)} \qquad \text{,,} \quad e'_\lambda\mathfrak{h}'_\lambda \quad (24)$$

$$e_{\lambda+1}\mathfrak{h}_{\lambda+1}=e'_\lambda\mathfrak{h}'_\lambda-t_\lambda(1-\cos^2\psi_\lambda) \qquad \text{,,} \, e_{\lambda+1}\mathfrak{h}_{\lambda+1} \quad (25)$$

$$\mathfrak{h}^2_{\lambda+1}=\mathfrak{h}'^2_\lambda-2e'_\lambda\mathfrak{h}'_\lambda t_\lambda+t_\lambda^2(1-\cos^2\psi_\lambda) \qquad \text{,,} \quad \mathfrak{h}^2_{\lambda+1} \quad (26)$$

$$d_\lambda=t_\lambda\cos\psi_\lambda-\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}'^2_\lambda)}+\sqrt{(\mathfrak{H}^2_{\lambda+1}-\mathfrak{h}^2_{\lambda+1})} \text{,,} \quad d_\lambda \quad (14)$$

It is evident on comparing these equations with those obtained in the earlier Paper that the conditions imposed have somewhat lengthened the calculations. The addition will, however, be found to be very slight. The appearance of the equations gives a misleading impression of complexity, but this is simply due to the notation employed. Two square roots have to be extracted to determine $\cos\varphi$ and $\cos\varphi'$, the same number as in the earlier system. All else is multiplication, division or addition. The trigonometrical equations of Von Seidel, which in appearance are remarkably simple, will be found to take up much more time. The latter involve at least 20 references to tables for each refracting surface, and the calculations when completed afford less information than can be derived from calculations on the algebraic system.

The earlier Paper already mentioned discussed the calculation of radial focal lines. It may be of interest to add the corresponding arrangement which has been found most convenient in dealing with the primary or transverse focal lines. As in the case of the radial focal line, the arrangement of the work resembles that followed for paraxial rays. Put

$$Y_\lambda=K_\lambda/\cos\varphi_\lambda\cos\varphi'_\lambda \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and let $\mathfrak{y}_{1,\lambda}$, $\frac{\partial \mathfrak{y}_{1,\lambda}}{\partial \mathfrak{y}_{\lambda}}$ be defined by the equations,

$$\left.\begin{aligned} \mathfrak{y}_{1\,\lambda} &= \frac{\cos \varphi_{\lambda}}{\cos \varphi_{\lambda}'} \mathfrak{y}_{1\,\lambda-1} + \mathfrak{y}_{\lambda} \frac{\partial \mathfrak{y}_{1.\lambda}}{\partial \mathfrak{y}_{\lambda}} \\ \frac{\partial \mathfrak{y}_{1,\lambda+1}}{\partial \mathfrak{y}_{\lambda+1}} &= \frac{\cos \varphi_{\lambda}'}{\cos \varphi_{\lambda}} \frac{\partial \mathfrak{y}_{1,\lambda}}{\partial \mathfrak{y}_{\lambda}} - \frac{d_{\lambda}}{\mu_{\lambda}} K_{1\,\lambda} \end{aligned}\right\}, \quad . \quad . \quad . \quad (28)$$

where

$$\frac{\partial \mathfrak{y}_{1,1}}{\partial \mathfrak{y}_{1}} = 1, \ \mathfrak{y}_{1,1} = \mathfrak{y}_{1}, \ \mathfrak{y}_{1,0} = 0.$$

The same value of $\mathfrak{y}_{1,n}$ is obtained by starting the calculations at the other end of the system, using the formulæ,

$$\left.\begin{aligned} \mathfrak{y}_{\lambda,n} &= \frac{\cos \varphi_{\lambda}'}{\cos \varphi_{\lambda}} \mathfrak{y}_{\lambda+1,n} + \mathfrak{y}_{\lambda} \frac{\partial \mathfrak{y}_{\lambda\,n}}{\partial \mathfrak{y}_{\lambda}} \\ \frac{\partial \mathfrak{y}_{\lambda-1,n}}{\partial \mathfrak{y}_{\lambda-1}} &= \frac{\cos \varphi_{\lambda}}{\cos \varphi_{\lambda}'} \frac{\partial \mathfrak{y}_{\lambda.n}}{\partial \mathfrak{y}_{\lambda}} - \frac{d_{\lambda-1}}{\mu_{\lambda-1}} \mathfrak{y}_{\lambda,n} \end{aligned}\right\}. \quad . \quad . \quad . \quad (29)$$

where
$$\frac{\partial \mathfrak{y}_{n,n}}{\partial \mathfrak{y}_{n}} = 1, \ \mathfrak{y}_{n,n} = \mathfrak{y}_{n}, \ \mathfrak{y}_{n+1\,n} = 0.$$

The quantities defined by these equations have properties analogous to the corresponding paraxial functions. Rays arising from a point at a distance p_0 along the incident ray in the first medium measured in the direction in which the light travels from the point at which the first refraction takes place will be refracted through a point on the emergent ray at a distance, p_n, from the point of refraction in the last surface measured in the same direction, where

$$p_0 = \mu_0 \left(- \frac{\cos \varphi_1}{\cos \varphi_1'}, \ \frac{\partial \mathfrak{y}_{1,n}}{\partial \mathfrak{y}_{1}} + \frac{1}{\mathfrak{g}} \right) / \mathfrak{y}_{1,n} \quad . \quad . \quad . \quad . \quad (30)$$

and

$$p_n = \mu_n \left(\frac{\cos \varphi'}{\cos \varphi_n} \ \frac{\partial \mathfrak{y}_{1\,n}}{\partial \mathfrak{y}_{n}} - \mathfrak{g} \right) / \mathfrak{y}_{1,n}, \quad . \quad . \quad . \quad . \quad (31)$$

provided the rays considered lie in an axial plane and lie close to the original ray throughout their passage through the optical system. Also if a small linear object lying in an axial plane and normal to the original ray meets the ray in the point distant p_0 from the first surface, its image formed by neighbouring rays in an axial plane will be normal to the emergent ray, will meet this ray in the point determined by p_n, and the linear magnification will be $\mathfrak{g}$.

Many other formulæ can be contrived for these calculations, and it is easy to put down generalised formulæ. It will be found that none of these are so convenient as those given above for general use, though in special cases they may offer distinct advantages. As an illustration, take the formulæ,

$$P_\lambda = K_\lambda$$

$$P_{\ ,\lambda} = \cos^2 \varphi_\lambda P_{1\,\lambda-1} + P_\lambda \frac{\partial P_{1,\lambda}}{\partial P_\lambda},$$

$$\frac{\partial P_{1\,\lambda+1}}{\partial P_{\lambda+1}} = \cos^2 \varphi'_\lambda \frac{\partial P_{1,\lambda}}{\partial P_\lambda} - \frac{d_\lambda}{\mu_\lambda} P_{1,\lambda},$$

with the symmetrical reverse formulæ,

$$P_{\lambda,n} = \cos^2 \varphi'_\lambda P_{\lambda+1\,n} + P_\lambda \frac{\partial P_{\lambda\,n}}{\partial P_\lambda}$$

$$\frac{\partial P_{\lambda-1,n}}{\partial P_{\lambda-1}} = \cos^2 \varphi_\lambda \frac{\partial P_{\lambda,n}}{\partial P_\lambda} - \frac{d_{\lambda-1}}{\mu_{\lambda-1}} P_{\lambda,n}.$$

It may be shown that

$$P_{1,n} = \mathfrak{M}_{1,n} \cos \varphi_1 \cos \varphi_1' \cos \varphi_2 \cos \varphi_2' \ldots \cos \varphi_n \cos \varphi'_n,$$

where the continued product on the right consists of the cosines of all the angles of incidence and emergence. Also

$$\cos^2 \varphi'_n \frac{\partial P_{1,n}}{\partial P_n} / P_{1,n} = \frac{\cos \varphi_n}{\cos \varphi_n} \cdot \frac{\partial \mathfrak{M}_{1,n}}{\partial \mathfrak{M}_n} / \mathfrak{M}_{1,n}.$$

Thus the terms involving the magnification and its reciprocal in the formulæ for conjugate distances would, with the P formulæ, require to be multiplied by the above continued product of cosines, and the labour of calculation in a complex system would be much increased.

It follows from the formulæ for conjugate distances that $\mathfrak{M}_{1,n}$ is the power of the system for a pencil of primary rays lying close to the given ray, just as $K_{1,n}$ is the power for a pencil of secondary rays. In the application of the formulæ for the primary rays it must be remembered that $\mathfrak{G}$ in general will differ from G for the same object and image points in a system corrected for astigmatism, because in the case of the former the dimensions of the object and image are not usually measured in planes normal to the rays. The appropriate value for $\mathfrak{G}$ is to be found by orthogonally projecting the actual dimensions of the object and image on to planes normal to the incident and emergent rays.

XIX.—*A New Method of Measuring Alternating Currents and Electric Oscillations.* *By* I. WILLIAMS, *M.Sc., Lecturer in Physics in the University of Bristol.*

RECEIVED APRIL 26, 1918.

§ 1. Alternating currents may be measured by means of the electromagnetic, the electrostatic, or the thermal effects which they produce. At low frequencies all these effects can be used, but when the frequency is high the electromagnetic effect cannot be usefully applied on account of the excessive dielectric currents, the values of which are functions of the frequency. Up to the present the application of the thermal effect has proved the most satisfactory basis for the construction of instruments to be used for the measurement of currents of the frequencies which are in common use in wireless telegraphy. If the resistance of the conductor which is heated by the current remains constant, the rate of generation of heat is strictly proportional to the square of the current. The rate at which the heat is generated has been measured by observing (*a*) the expansion, (*b*) the change of electric resistance, or (*c*) the thermoelectric electromotive force which is produced in the wire itself or in its immediate surroundings.

In this Paper a new thermal method of measuring alternating currents and electric oscillations is proposed, and two new types of instrument are described. Type 1 is based on Crookes'* well known work on radiometers ; the heat generated by the currents in a wire causes the deflection of a mica vane suspended in a vessel, the pressure in which can be maintained at a suitable value. Type 2 is based on work done by Osborne Reynolds † ; the mica vane is replaced by a filament or narrow strip of suitable material, the deflection of which can be observed directly by means of a microscope. Type 1 may be said to resemble an electrometer, while Type 2 resembles an ordinary gold-leaf electroscope.

The method proposed has not, so far as the author is aware, been previously applied to the measurement of alternating currents and electric oscillations. When the application of this method of measurement was first considered it was thought quite possible that the attractions, which are produced at certain pressures, might be as useful, or perhaps, more useful than the repulsion effects produced at lower pressures. For this reason it was considered advisable to investigate the relative magnitudes of these quantities from atmospheric pressure down to very low pressures.

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* "Phil. Trans.," Vol. CLXV., p. 579, 1875.

† "Phil. Trans.," Vol. CLXX., p. 768, 1879.

§2. The first form of apparatus used is shown in vertical section in Fig. 1. *ABCD* is a circular brass casting which is flanged at *AD*, and supported on three levelling screws. In this casting three holes were drilled, one of which is shown at *E*; the other two were situated on a line, through the vertical axis, perpendicular to the plane of the paper. The mirror *R*

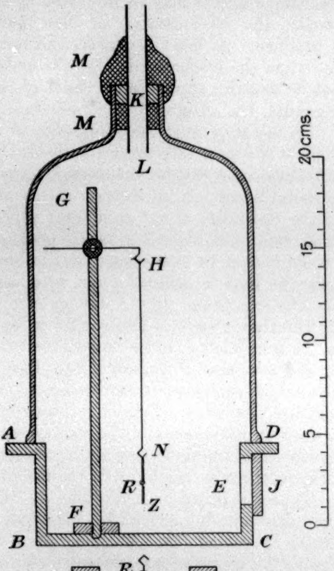


Fig. 1.—High-frequency Ammeter. Type 1.

could be observed through a plate glass window *J* and the heaters could be inserted through the other two holes. *FG* is a rod carrying a hook *H*, from which the vanes were suspended, and which could easily be adjusted to any suitable position. Resting on the casting and cemented to it is a bell-jar, in the neck of which was fixed an ebonite stopper *K* through which passed a glass tube *L*; the stopper was covered internally and externally with "Picein" at *MM*. *L* was connected to the pump and the McLeod gauge by a flexible glass connection which allowed just sufficient freedom for levelling.

The vanes *P* and *Q*, which were each about 15 mm. square, were made of very thin mica and were blackened on the faces exposed to radiation from the heated wire; they were suspended from the hook *H* by a quartz fibre which was about 10 cm. long. *XY* is a very thin glass capillary and *Z* is a very small light magnet; the hooked mirror carrier was made of aluminium. The heaters were made of a nickel-chrome alloy. Two heaters were provided so that they could be made to deflect the vanes in the same, or in opposite, directions.

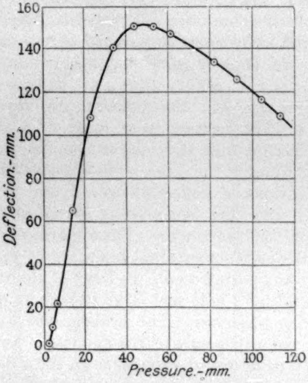
The observations of deflection were made by means of a telescope and scale, the latter being placed at a distance of about 120 cm. from the mirror. The apparatus was protected from draughts and from external radiation by means of cardboard screens and cotton wool. In front of the window *J* was a screen pierced by a hole just large enough to permit of the deflection being observed. The heating coil was connected to earth and the upper part of the apparatus was earthed, as far as possible, by layers of wire gauze on the outside.

The high-frequency current can be measured (1) by observing the deflection and using a calibration curve obtained with direct current, or (2) by a Null method. In the second case, a first balance is obtained when the same direct current of known value is passed through the two heaters in series, in such a direction as to tend to deflect the two vanes in opposite directions; then, a second balance is obtained when the high-frequency current flows through one heater and the direct current through the other heater is adjusted until the vane is again in the same position.

§ 3. The observations over the pressure range 1·45 mm. to 113 mm., the value of the heating current being 0·5 ampere, are recorded in Table I. and plotted in curve 1. The attraction rose rapidly to its maximum value at about 45 mm., and then decreased more slowly over the remainder of the range.

TABLE I.—*Mica Vane A. Attraction.*

Pressure. Millimetres.	Deflection. Millimetres.	Pressure. Millimetres.	Deflection. Millimetres.
1·45	0·0	44·0	150·5
2·08	3·0	53·0	150·0
4·08	10·0	61·0	147·0
6·08	21·5	82·0	133·5
14·0	64·5	93·0	126·0
23·0	108·5	104·0	116·5
34·0	140·5	113·0	109·0



CURVE I.

The heating current was now varied from 0·1 ampere to 0·5 ampere, while the pressure was maintained constant at 33 mm. The observations indicated that the deflection of the vane was very accurately proportional to the square of the current.

§ 4. When observations were made at lower pressures leakage troubles arose, and as it was suspected that the leak took place at the circular surface of separation between the brass casting and the bell-jar, the form of apparatus described above was abandoned for the present in favour of a form constructed entirely of glass, a vertical section of which is shown in Fig. 2.

ABCD is a hollow cylinder which has ground flanges at each end and ground joints in the vertical tubes *E* and *F*. The vane suspension passed through *E* and the leads to the heater through *F*. The end *AB* was closed by a piece of plate-glass 0·5 in. thick and the end *CD* by a bell-shaped tube *CDP*. The joints at the ends *AB* and *CD* were cemented with "Picein."

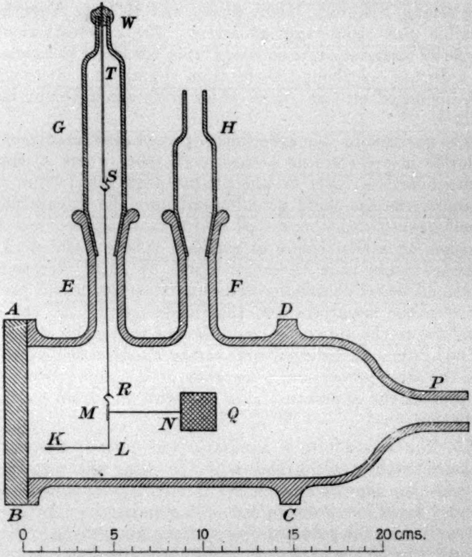


FIG. 2.—HIGH-FREQUENCY AMMETER. TYPE 1.

The vane *Q* was a rectangular plate of very thin mica, 20 mm. × 20 mm., blackened on the side exposed to radiation from the heated wire. *K* was a damping plate made of very thin copper foil, which could move over the pole of a magnet not shown in the figure. *KL* and *MN* were short light glass capillaries, while *RML* was an aluminium wire to which the mirror *M* was attached.

The heater was a short straight length of platinum wire which had a resistance of about 0·9 ohm. The two leads passed out through the ground joint in *F* and were insulated from one another by glass tubing. The heater was kept fixed while the vane could be adjusted to any suitable distance from it by rotating the tube *G*.

The whole arrangement was mounted on a levelling table, which stood on a brick pillar. The end *P* was connected to the storage chamber, Gaede pump, and McLeod gauge by a flexible glass connection as before. The observations were made by means of a telescope and scale, the latter being placed at a distance of about 120 cm. from the mirror. Everything was mounted on slate slabs which were supported by brick pillars.

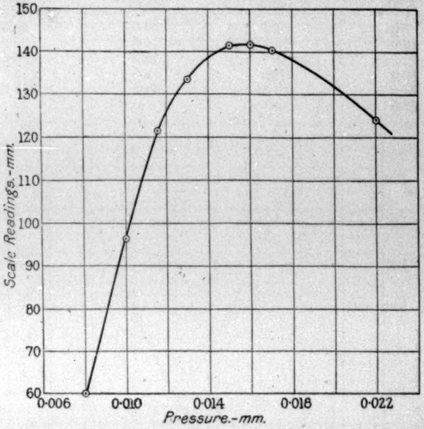
The gas used for the experiments passed over fused calcium chloride before entering a storage chamber where it was in contact with a layer of phosphorus pentoxide; from this chamber the gas could be admitted into the apparatus in small quantities by means of two taps separated from one another by a short piece of capillary tubing; the working pressure could thus be easily varied by small increments. A second vessel containing phosphorus pentoxide was placed between the apparatus and the Gaede pump. All the gas admitted to the apparatus could thus be thoroughly dried.

The heater and the vessel were earthed, as already described, and the same precautions were taken, as in the previous case, to protect the apparatus from draughts and from external radiation.

§ 5. The second form of apparatus was quite free from the leakage troubles which had arisen in using the first form. Dry air was repeatedly pumped through the apparatus and the dry air at low pressure had been standing in it for about a week before the recorded observations were made. When the pressure in the apparatus had been reduced to a suitable value by the Gaede pump and an electric current which remained absolutely constant was passed through the heating coil the reading on the scale was observed; a suitable quantity of dry gas was then admitted from the storage chamber and the scale reading again observed, and so on. The results obtained for the pressure range 0·008 mm. to 0·022 mm. are given in Table II. and curve 2 for a current of 0·1 ampere. The repulsion reached its maximum value at a pressure of about 0·0155 mm.

TABLE II.—*Mica Vane B. Repulsion.*

Pressure. Millimetres.	Scale reading. Millimetres.	Pressure. Millimetres.	Scale reading. Millimetres.
0·008	60·0	0·015	141·2
0·010	96·3	0·016	141·2
0·0115	121·5	0·017	140·2
0·013	133·5	0·022	124·0



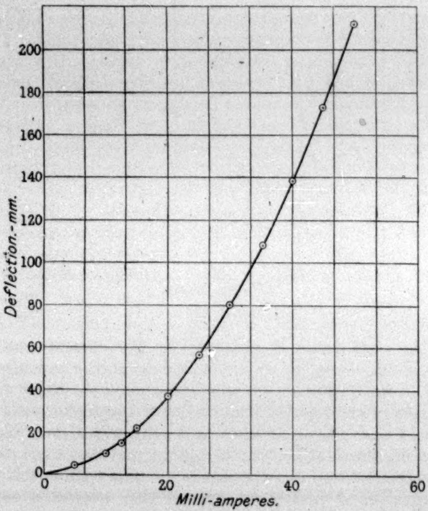
CURVE II.

In the next series of observations the current was kept constant and equal to 0·077 ampere while the pressure was varied from 0·020 mm. to one atmosphere in order to get some idea of the relative magnitudes of the repulsions at low pressures and the attractions at higher pressures. The repulsion decreased at first fairly rapidly, and then more slowly, until it became zero at a pressure of 4·6 mm. From this point the repulsion at first increased and then decreased to zero again at 288 mm. Over the pressure range 288 mm. to one atmosphere the vane was subject to an attraction which increased in value up to atmospheric pressure. As the apparatus was designed for work at pressures below one atmosphere no observations were made at pressures greater than atmospheric.

§ 6. The pressure was now reduced to 0·014 mm. and the current varied from zero to 50 milliamperes. The values obtained are given in Table III. and curve 3. Table IV. and curve 4 indicate the results obtained, for a pressure of 0·016 mm.

TABLE III.—*Mica Vane B.*

Current. Milliamperes.	Deflection. Millimetres.	Current. Milliamperes.	Deflection. Millimetres.
5·0	4·2	30·0	80·5
10·0	10·2	35·0	108·2
12·5	15·0	40·0	139·0
15·0	22·0	45·0	173·5
20·0	37·5	50·0	212·0
25·0	56·5	—	—

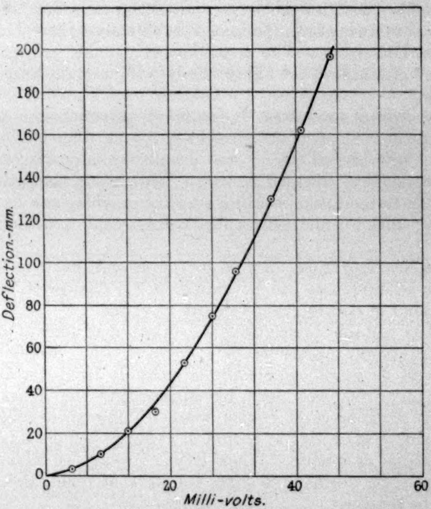


CURVE III.

when the instrument was used as a millivolt meter. A current of 5 milliamperes gave a deflection of 4·2 mm., whilst a potential difference of 4·4 millivolts gave a deflection of 3·8 mm.

TABLE IV.—*Mica Vane B.*

Potential difference. Millivolts.	Deflection. Millimetres.	Potential difference. Millivolts.	Deflection. Millimetres.
4·4	3·8	26·8	75·3
8·9	10·3	30·5	96·0
13·2	21·0	36·0	129·8
17·6	30·0	40·8	162·3
22·2	53·0	45·6	197·0



CURVE IV.

§ 7. The same heater was used in carrying out the experiments, the results of which are given in Tables II., III. and IV. The tables and curves indicate that the maximum deflection, for the apparatus used, is obtained at a pressure of about 0·015 mm. The deflection due to repulsion at this pressure is roughly about six times as great as the deflection due to attraction at atmospheric pressure. Curves 3 and 4 indicate that the instrument is quite satisfactory as an ammeter or voltmeter. The sensitiveness for currents which is indicated by curve 3, viz., 4·2 mm. for 5 milliamperes can obviously be increased to a very considerable extent. The resistance of the heater used was only 0·9 ohm. Since the heating effect is proportional to the product of the resistance and the square of the current the sensitiveness can obviously be raised by using heaters of higher resistance, and experiments are now being carried out with a view to designing a suitable series of different magnitudes.

The fibre used for the suspension of the vanes was a home-made one, and later experiments have shown that this fibre was rather coarse. The sensitiveness can be increased by replacing this fibre by a fibre which is thinner and somewhat longer.

The mirror used was an ordinary galvanometer mirror 10 mm. in diameter and weighing about 100 milligrams. The weight of this mirror was a very considerable fraction of the weight of the moving parts. Here again the sensitiveness can be increased by using a mirror weighing say 20 milligrams ; such mirrors are supplied commercially at a moderate price.

The instrument is suitable for large and small currents, as the sensitiveness can be varied by (1) change of heater, (2) change of control, or (3) change of vanes.

Unless suitable precautions are taken, instruments of this type are subject to considerable changes of zero. It is essential to shield them as completely as possible from extraneous radiation ; this was done by surrounding the instrument with a thick layer of cotton wool, leaving uncovered just sufficient window area to permit of readings being made with a telescope and scale. When it is necessary to illuminate the scale, by means of a lamp, for work at night, the lamp must be screened.

Dellinger* has discussed very fully the arrangement of circuits in high-frequency ammeters having one or more hot wires. If one hot wire or strip is used the leads should be brought in perpendicular to the ends of the hot wire. For convenience in carrying out the present experiments the leads have not actually been introduced in this way, but the apparatus can easily be modified slightly to permit of this being done. The shape of the apparatus shown in Fig. 2 is a matter of accident ; this particular glass vessel, which had been constructed for other work, was immediately available and was used because of the difficulty in getting new apparatus constructed in reasonable time. In the apparatus now in course of construction, the high-frequency current enters and leaves at two points which are a considerable distance apart ; the mass and the moment of inertia of the moving system have also been considerably reduced. In discussing branch circuits Dellinger (*loc. cit.*, page 93) points out that " when the indicated current depends on the heat production in just *one branch* (of a multi-wire instrument) the error due to change of current distribution on high frequencies may be very great. When the indicated current depends on the heat production of the *whole* current, the error will be of a smaller order of magnitude, but may be appreciable when the change of current distribution is great, inasmuch as the total heat production in any system increases as the distribution on direct current is departed from." In the method of current measurement proposed in the present Paper the heat production of the whole current is utilised, and the errors which arise are thus kept as low as possible.

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* "Bulletin," Bureau of Standards, Vol. X., p. 91, 1914.

§ 8. Experiments have also been carried out with an apparatus of the electroscope form, referred to above as Type 2, and shown in Fig. 3. It was shown by Osborne Reynolds * in his discussion of the movement of Crookes' vanes and allied problems that a silk fibre suspended in an exhausted tube is subject to forces of the same nature as those acting on the vanes of a radiometer. Reynolds also showed experimentally that when the silk fibre was exposed to radiation from an external source it was repelled over the pressure range 0·6 mm. to 200 mm. and attracted over the range 200 mm. to one atmosphere. It seemed clear, that if the deflection of the Crookes' vanes could be used for current measurement, the motion of a fibre could also be used for the same purpose.

A silk fibre *AB* about 75 mm. long attached to the end *A* of a metal rod *AC* which could slide up and down, rotate about, or be moved to or from the vertical rod *CD*. The latter was mounted on an ebonite block *DEF* and connected to one terminal *G* and also to earth. Starting from the terminal *G*, the heated wire *GH* passed over a hook at *H*, over two more hooks not shown, and finally was clamped to the terminal *G′* immediately behind *G*.

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* " Phil. Trans.," Vol. CLXX., p. 768, 1879.

The ebonite block *DEF* with its heating coil and fibre was mounted in the lower half of a wide vacuum tube *XYZ*, the leads to the terminals at *GG′* being cemented in at *Z* and insulated from one another by a glass tube. The end *X* of the tube was permanently connected to the McLeod gauge, storage chamber, and Gaede pump, as in previous work.

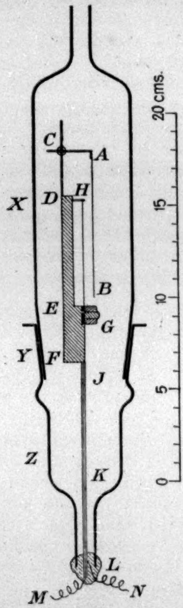


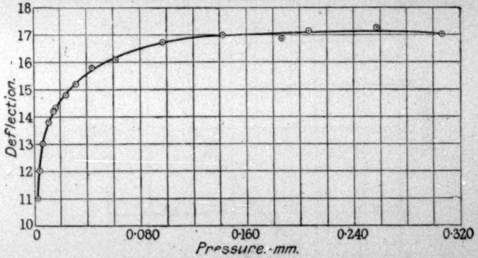
Fig. 3.

By means of the ground joint at *Y*, the whole arrangement could be taken out and the fibre or the heating wire changed or adjusted. When a current was passed through the wire *GH* the motion of the end *B* of the fibre was observed by means of a microscope having an eyepiece scale. The microscope available for the work was not a particularly good one, as the magnification was very low, being only about nine.

§ 9. The apparatus was pumped out repeatedly, as in the previous work, and then filled with dry air from the reservoir. Before the observations recorded in the tables were made the dry gas had been standing in the apparatus for some days. The pressure was reduced as far as possible and the deflection of the fibre was then observed for various pressures from 0·002 mm. to 2·94 mm., the current being kept constant and equal to 0·05 ampere. These observations are given in Table V.

Table V.—*Silk Fibre. Repulsion.*

Pressure. Millimetres.	Deflection.	Pressure. Millimetres.	Deflection.
0·002	11·0	0·186	16·9
0·004	12·0	0·206	17·1
0·0065	13·0	0·258	17·2
0·011	13·8	0·306	17·0
0·016	14·3	0·39	16·8
0·015	14·2	0·48	16·3
0·024	14·8	0·72	14·8
0·032	15·2	0·97	13·6
0·044	15·8	1·66	10·7
0·071	16·1	2·38	8·8
0·097	16·7	2·94	7·6
0·142	17·0	—	—

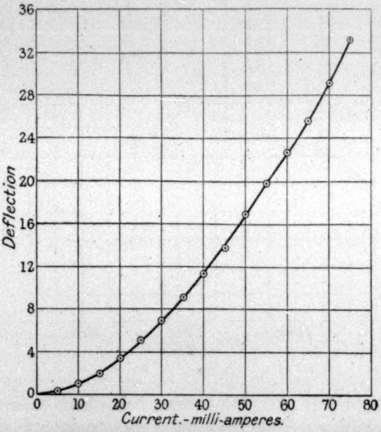


Curve V.

The zero was very steady and the deflection for a given current and a given pressure could be repeated over and over again. The return to zero was very rapid. The following numbers illustrate the behaviour of the fibre apparatus in this respect very well:—(1) Current of 50 milliamperes switched on gave a reading of 57·0; 15 seconds after break the reading was 73·5; and in 30 seconds the reading had reached 74·0 and remained steady. (2) Same current switched on again the reading was 57·0; 15 seconds after break it was 73·5; and in 30 seconds it was 74·0. (3) The same current was switched on a third time; the reading was again 57·0; 10 seconds after break it was 73·0; in 20 seconds it was 73·8; and in 25 seconds it was 74·0.

TABLE VI.—*Silk Fibre.*

Current. Milliamperes.	Deflection.	Current. Milliamperes.	Deflection.
5·0	0·3	45·0	13·7
10·0	1·0	50·0	16·9
15·0	1·8	55·0	19·8
20·0	3·4	60·0	22·7
25·0	5·0	65·0	25·6
30·0	6·8	70·0	29·2
35·0	9·0	75·0	33·2
40·0	11·2	—	—



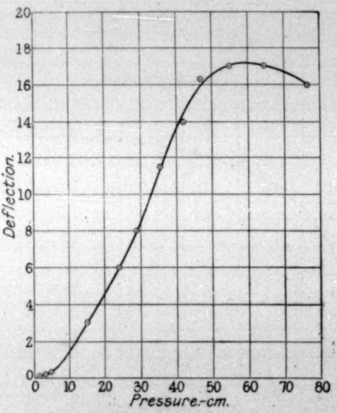
CURVE VI.

It is clear from Table V. and curve 5 that within the pressure range 0·002 to 3 mm., the repulsion reaches its maximum value at a pressure of about 0·28 mm. Between 0·1 mm. and 0·4 mm. the curve is very flat and obviously the instrument can be used as a current measurer at any pressure between these limits. That the maximum repulsion should be reached in the case of a thin filament at a much higher pressure than in the case of the vane is in accordance with Reynold's theoretical conclusions; the narrower the body the higher the pressure at which the repulsion effect is a maximum.

TABLE VII.—*Silk Fibre. Repulsion.*

Pressure. Centimetres.	Deflection.	Pressure. Centimetres.	Deflection.
1·4	1·0	35·5	11·5
3·1	1·3	42·0	14·0
4·7	1·4	47·0	16·3
15·0	3·0	55·0	17·0
24·0	6·0	64·0	17·0
29·0	8·0	76·0	16·0

Current = 0·05 ampere.



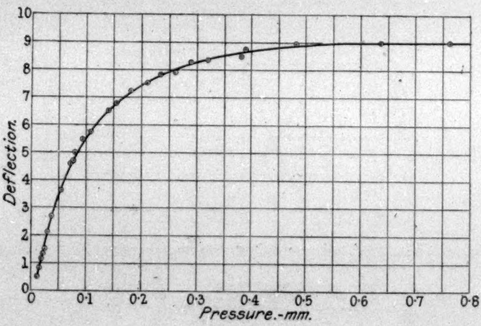
CURVE VII.

The pressure was next maintained constant at 0·28 mm., while the current was varied from 0 to 75 milliamperes; the current was measured with a Siemens milliammeter as before. The results are given in Table VI. and curve 6. As shown above the instrument possesses the following desirable qualifications for making a good ammeter: (*a*) steady zero and (*b*) rapidity in action.

TABLE VIII.—*Quartz Fibre B. Repulsion.*

Pressure. Millimetres.	Deflection.	Pressure. Millimetres.	Deflection.
0·010	0·5	0·142	6·5
0·015	0·8	0·156	6·8
0·017	1·1	0·180	7·2
0·020	1·3	0·210	7·5
0·025	1·5	0·235	7·8
0·031	2·1	0·26	7·9
0·041	2·7	0·29	8·3
0·053	3·6	0·32	8·4
0·069	4·6	0·38	8·5
0·074	4·7	0·39	8·8
0·080	5·0	0·48	9·0
0·094	5·3	0·64	9·0
0·110	5·7	0·76	9·0

Current=0·1 ampere.



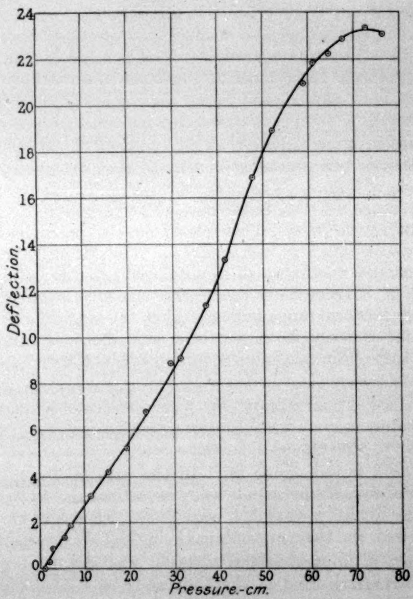
CURVE VIII.

Over the pressure range 1·4 mm. to one atmosphere, the deflection reached its maximum value at a pressure of about 60 cm. as shown in Table VII. and curve 7. These results

TABLE IX.—*Quartz Fibre B. Attraction.*

Pressure. Centimetres.	Deflection.	Pressure. Centimetres.	Deflection.
1·5	0·0	37·4	11·4
2·3	0·3	41·4	13·3
3·4	0·9	47·6	16·9
5·8	1·4	52·2	18·9
7·2	1·9	58·7	21·0
11·5	3·2	61·1	21·8
15·5	4·2	64·6	22·2
19·7	5·2	67·7	22·8
24·0	6·8	72·6	23·3
29·5	8·9	76·4	23·0
31·6	9·1	—	—

Current = 0·03 ampere.



CURVE IX.

differ from those of Reynolds in that a repulsion was observed with the silk fibre for all pressures from 0·002 mm. to one atmosphere. The deflection at atmospheric pressure is of the same order as that obtained at 0·28 mm. So far as one can judge at present the instrument is more steady in its behaviour at low pressure than at atmospheric pressure.

§ 10. *Glass Fibres.*—A number of fine glass fibres were drawn out and the thinnest one selected. This was mounted in the apparatus with the intention of making a similar set of determinations for glass; but the fibre was too coarse and stiff and the deflections were too small to be dealt with by the microscope available. With a microscope of higher power there would be no difficulty in using glass fibres.

§ 11. *Quartz Fibres.*—The quartz fibre *A* used for the early experiments was home-made. It was rather coarse and stiff, but a complete set of observations was obtained by means of it. This fibre was replaced by fibre *B* made by the Cambridge Scientific Instrument Co., and the observations given in Tables VIII. and IX. and curves 8 and 9 are those obtained with fibre *B*.

The observations showed that the variation of the deflection with pressure was somewhat similar to that observed in the case of the silk fibre. The curve was practically flat over the pressure range 0·4 mm. to 0·8 mm. The repulsion decreased to zero at a pressure of 1·5 cm., and from this point to atmospheric pressure the fibre was attracted. The attraction seemed to approach a maximum at atmospheric pressure, as shown in curve 9. Observations made with the same current over the whole pressure range indicated that the magnitude of the attraction, at atmospheric pressure, was about 16 times as great as that of the maximum repulsion over the lower pressure range.

In the case of quartz, a deflection current-curve is not given in this paper, as some irregularities arose the sources of which have not yet been traced definitely.

This work is still going on. In the case of Type 1 the question of suitable heaters is being investigated. As regards Type 2 preliminary work has been done with spider threads and gold leaf, and the quartz fibre is being further investigated.

The cost of apparatus and material was met by a grant from the research fund of the University of Bristol Colston Society.

ABSTRACT.

The method consists of the application of the Crookes and Osborne Reynolds radiometers to the measurement of the R.M.S. values of electric currents. Two types of apparatus are descirbed. In the first of these the heat generated by the passage of the current through a nichrome resistance causes the deflection of a light mica vane attached to the extremity of a suspended beam. In the second type the deflection of a fine fibre is employed. Tables and curves are given connecting the indications of the instruments with the current and with the degree of evacuation.

DISCUSSION.

Mr. G. D. WEST said that in his Paper on the Measurement of the Pressure of Light by the deflection of a thin metallic strip he had called attention to certain disturbances due to gas action. He treated them at that time as sources of error, but had since been engaged on their investigation. He could not go into his results here, but would just like to say that some of the effects shown by the author were not wholly explicable by the Crookes radiometer effect. He would also like to point out that Knudsen had studied the deflection of strips placed close to an electric heater. There were possibilities of the useful application of some of these effects when they were properly understood. At present, however, there was considerable uncertainty in connection with some of them.

Dr. SUMPNER thought the instrument described by Mr. Williams was likely to prove a valuable one. Comparatively little information was given concerning the heaters, so it was difficult to judge of their suitability for the currents employed in wireless telegraphic work, which were very much smaller than those mentioned. There was the danger that if the resistance had to be made too high the instrument would alter the magnitude of the current it had to measure.

Prof. HOWE said that one of the difficulties in measuring high-frequency oscillations was the effect of electrostatic forces between the heater and the thing heated. It appeared to him that this type of instrument was particularly liable to this source of disturbance. An obvious way to overcome this was to put a dummy heater on the other side of the vane or fibre and put it in electrical connection with the heater. He thought that in its present form the instrument could not be very convenient in use

Dr. D. OWEN said he was not certain that the difficulties inherent in the measurement of minute forces had all been surmounted. From Table I. it would appear that they had, as in this case the deflections were stated to be accurately proportional to C^2. This law was not, however, followed in Table III., the deflections being all smaller than they should be, despite the fact that a platinum heater was used in this case, which would lead one to expect too high readings. He observed that the author connected the heater to earth, and gathered from this that electrostatic disturbances had been noticed. It was not easy to apply this method of elimination of these effects in the case of high-frequency oscillations.

Dr. W. ECCLES said that some years ago he had built a little apparatus which would detect a microampere. The heater was a short glass fibre platinised. It was mounted about 2 mm. below the extremity of a light horizontal fibre of quartz, which constituted the beam of a delicate micro-balance. The effect was best at atmospheric pressure, and he had always attributed it to convection currents; but it seemed that there might be many other factors entering into it. The heater was of about 1,000 ohms, so that the value of the watts consumed per micro-ampere was 10^{-9}. He also made another type, in which a light vane was mounted at the end of the beam. This was several times as sensitive as the first. He had described these in one of his Papers at the time.

Prof. **Lees** asked how the pressure was maintained sufficiently steady, as the sensitivity appeared to vary considerably with the pressure.

Mr. **Williams**, in reply, said he had had no difficulty with pressure. The change of sensitivity with pressure was small near the top of the curve, at which he usually worked. This question did not, of course, arise at all when employing the Null method. He had not seen any of Knudsen's work when he started these experiments, and originally intended to use the instrument as a manometer, as it was suitable for measuring very small changes of pressure due to leakage or transpiration through substances. He subsequently found, however, that Knudsen's work already covered this ground. With regard to Dr. Sumpner's remarks on the magnitude of the current, it had to be borne in mind that the instrument described was an experimental one, and could be very much improved by reducing the inertia of the suspended system and using a much finer suspension. With regard to the square law, the graph in the Paper applied to the first type of instrument only; he thought Dr. Owen's remarks applied to observations made with the other type. As regards electrostatic disturbances, he thought at first he detected, effects of this kind, but found they were unaffected by the proximity of radium. He had subsequently found them to be mostly due to other causes.

An Exhibition of Coupled Vibrations was given by Prof. E. H. Barton, *F.R.S., and Miss* H. M. Browning.

The apparatus shown consisted of a pair of pendulums, each of which was suspended from the mid point of a sagging string, the direction of which was transverse to the direction of oscillation of the pendulums. The two sagging strings were connected by a light wooden rod at the points from which the bobs were suspended. Each bob consisted of a metal funnel, from the apex of which a fine stream of sand fell during an experiment. A horizontal board could be moved slowly on rails just below the oscillating bobs, and the fine sand falling on this gave curves showing their motion. When one bob is set in oscillation the other being initially at rest, the latter, as is well known, starts to vibrate with gradually increasing amplitude until the first bob has been brought to a standstill, when the process is reversed. From an examination of the equations of motion it is found that the amount of sag in the transverse strings governs the degree of "coupling" of the oscillators, and by varying this and also the relative mass and periods of the pendulums, curves can be obtained illustrating all the phenomena of coupled electrical oscillations. By stopping one of the bobs when it has just been reduced to rest, thereby preventing the energy from being reabsorbed by it, the conditions of the quenched spark can be imitated.

THE TEACHING OF PHYSICS IN SCHOOLS.

A MEETING of the Society was held on Friday, June 14, 1918, at the Imperial College of Science, South Kensington, when a discussion took place on " The Teaching of Physics in Schools." The chair was taken by the President, Prof. C. H. LEES, F.R.S.

The PRESIDENT said : This meeting is the outcome of a desire on the part of the Physical Society to help those of its members who are engaged in science teaching, especially those engaged in teaching in public and secondary schools, to carry out the probable extension of that teaching which will ensue in the course of the next few years. At the present time there is a great demand throughout the country that the average well-educated man shall know more physics in the future than he has known hitherto, and the physics teachers will have to play their part in furthering that advance. With these explanatory remarks, I will call upon Sir Oliver Lodge to open the discussion.

Sir OLIVER LODGE, Principal of the University of Birmingham, said : Mr. President, I very much agree with your observations that it is desirable that the average man shall know more physics than he does at present. He could hardly know less. But I do not quite know why I should have been selected to open the discussion (except that it may be thought that I should at any rate not close it), because there was no teaching of science in schools when I was young. I hardly knew what science was until I had left school, and I never took part in school teaching. I have taught in universities all my life, but university teaching is comparatively easy, and is outside our subject this afternoon. What I am impressed with is that the teaching of science in schools is a very difficult task. I have read through the Papers which have been circulated before the meeting, and I think there is a large amount of wisdom in them. I should suppose that any teacher who seriously set to work to think about the teaching of his subject, and was keen enough to read a Paper on the methods of teaching it, was likely to be a good teacher. I expect there are lots of methods of teaching, and, as Kipling would say, " every single one of them is right," provided they are the outcome, not only of knowledge, but of enthusiasm for the subject. That seems to me to be about the most essential part of the business. Curricula are minor things in comparison, and yet they must exist, and no doubt some curricula are better than others.

Now, in speaking to an audience of specialists it is unnecessary to be elaborate or long, and since there are many speakers we shall all have to be short ; I will try to set a good example. In the first place, we must discriminate between the teaching of science to specialists, or to those who are going to be specialists in the subject, and, on the other hand, the teaching of science as a branch of general education. The first is much easier than the second. If you are teaching boys who have a liking for physics, almost any method of teaching will do. But if you are teaching science as a branch of general education to those who are going to be classical scholars or literary people, or people not specially scholars at all, then you have to interest them in the subject—they may not be naturally interested—and you must avoid making it repulsive to them. And I think that what you need from the point of view of general education, which after all is required by the majority of boys, is to take a wide scope—a wide survey, without emphasising too much the details, or, rather, emphasising the details precisely and accurately in a small branch of the subject, and covering a much greater area widely and generally. The same sort of thing is necessary in classics, too ; the hole-and-corner strict grammatical method without any key or note was a severe grind and discipline, but it did not teach one literature ; it was merely a language grind. That is proper and good up to a point, but it should not be the whole ; it is not a classical education. The large survey of literature, whether in translations or otherwise—in translations whenever necessary—should be employed as well ; and that represents, in a sort of parable, what is true also of science.

I should like to call attention to the recent report of the British Association Committee on science teaching, which, I suppose, is well-known to all of you ; there are some excellent Papers in that report, but there are one or two—especially one at the end—with which I to some extent find myself in doubtful agreement, or, I might say, in disagreement, except that I am not an expert in school teaching. And yet there is a sort of truth about it, too. The thesis is that you should begin by dealing with common objects. Take any common object and analyse it down to principles, do not begin with the principles and build up. That is to say, employ the method of analysis—the analytic method rather than the synthetic method. I think that the wise policy is to run the two together, but it depends a great deal upon whether you are dealing with adults or with children. If you are dealing with adults, as Prof. Perry was, for instance, in the evening classes at Finsbury, you may do as he did, and take a gas engine or a steam engine or anything to which the men are accustomed in their daily work, and analyse out the principles of it, working down to thermodynamics from the engine it you can. That method is possible with adults, though I think difficult, and requiring some genius in the teacher, but it is unsuitable for boys. There you must begin with elementary principles and build up; not, however, keeping the steam engine in the background—let them attend to that as much as they like, but do not make it the basis of your education. And what I have said about the steam engine applies to all sorts of common objects. What I feel about common objects is that they involve too much mathematics. They are too difficult. Take the blue of the sky, for instance. The blue of the sky occurs to me because of a brilliant Paper by Prof. Strutt, verifying his father's theory of the blue of the sky by some beautiful experiments. It is quite right to give the child some idea of why the sky is blue, but if you make it a systematic part of the teaching you will have to give him Rayleigh's theory about how it happens through the discontinuity of the atoms, and you will lead him into great depths. Or, take a simple thing like the shape of a chain; hang up a rope—a skipping-rope—and ask, What is the shape of that? It is an excellent example for geometry. The properties of the cycloid or the catenary are very interesting, but they are not easy. You may even treat it as an easy problem in the Calculus of Variations, to find out what curve has its centre of gravity lowest; and it comes out the catenary. But all these things are more interesting when you know mathematics than when you don't. Or, take the spinning of a top—a whip-top—a familiar object, but to explain it properly you need Sir George Greenhill or somebody of that kind.

But it may be said, What about the materials which, while quite common, do not obviously involve mathematics, such as those dealt with in the Paper I spoke of, where attention is called, for instance, to the smeariness of butter. You are supposed to take butter in the fingers and feel the smeariness of it, and you might similarly deal with the stickiness of jam. What I say about those things is that they also are an adult study, in so far as they are a study at all. The child knows these things; takes them for granted: it does not occur to him to treat them scientifically; that would only occur to adults. When you have done a lot of physics and chemistry then you may recognise the science underlying common things. Then you may perceive that viscosity is interesting, and may be able to deal scientifically with many common experiences; but I do not feel that treacle, butter and ordinary things are interesting to the young; or that their powers of observation can well be cultivated upon them.

For cultivating the faculty of observation the biological sciences are surely the best to begin with; and for examples which involve mathematics I should take simple things, artificial things, things which can readily be dealt with, and dealt with graphically. One such thing would be, having drawn an ellipse, to tell the students to find out some things about it to, verify the equality of the angles, for instance—the principles which enter into the making of a mirror. Or, take any other bi-polar curve: Instead of taking the sum of the distances constant to two fixed points, take the difference, or the product, or the ratio, of the distances as constant. Thus, draw two points 3 in. apart, and make a curve every point of which shall be twice as far from one as the other. Well, you get a circle, and if you take different ratios you get a system of co-axial circles; but the boy will not expect that. He will make a drawing, and if he draws it right, you can easily check it; that is one advantage of problems for examination purposes. This kind of problem illustrates to the child the sort of law and order that runs through things. And it is more interesting when he discovers a fact for himself, and does it as an exercise, than when he is merely told about it. He need not be told how to draw a circle through these points, or round a triangle, for instance. Then I dare say something can be done with experiment on a pendulum, the plotting of simple harmonic motions, beats and interferences; and then, again, leaks, the leak of water out of a vessel, or of heat out of a body, or the leak of electricity from a condenser; the theory of leaks—the logarithmic decrement business—gives a good exercise at a higher stage.

Nearly all these things can be dealt with properly by a mathematical teacher. But there are certain problems in statics which seem very dull, and which, I think, can be enlivened by reading old books on mechanics. I came across recently Galileo's "Dialogues on Mechanics"—not the "Dialogues on the Copernican System," but on mechanics—they are translated into English—and the kind of problems treated are quite interesting. Then, there are Thomas Young's "Lectures on Natural Philosophy," which are full of instruction to a teacher. And there are Newton's experiments on the prism, the classical example of accurate experimenting in a new field. But what I feel is that teachers are very apt to overdo the metrical side of physics. It is easy to give people things to measure, but unless they take some interest in the things themselves they do not want to measure them, and the mere learning of how to measure them does not do any particular good. I used to put the students through certain courses of junior study, such as determining the height of a barometer, the time of a pendulum, the moment of a magnet, specific gravity, expansions, specific heat and many other things ; they went through these courses—or curses, as they used to be called—but I never felt quite happy about them. It seemed to me that they were rather dull (I was giving them lectures as well, of course, which were more lively). The great bulk of them did not want to measure the things at all, and it was very little use to show them how to measure things of no interest.

And remember how artificial measurement is ! An electric current was a fascinating thing in youth to me, but I never thought of expressing it in amperes. Amperes, volts and ohms are quite recent—at least more recent than I am. Things themselves, apart from units of measurement, are interesting ; and from the point of view of having a wide scope some acquaintance with a large number of real things is desirable, though how it is to be managed with any precision I do not know.

What I have in my hand are the notes of an article I have recently written, and which will appear in the August "Fortnightly Magazine"—why it is called the "Fortnightly," being a monthly, I cannot say—on Humanistic Science, that is to say, science treated as one of the humanities. I do not say it is easy so to treat it, but I do think we must make the attempt, and realise that for the majority of children science should be treated partly in that way. Literature, for instance, should not be treated simply by rules of syntax and prosody—you could spoil literature by over-attention to syntax, and spoil a poem by treating it as an example of prosody, and we are rather apt to treat science in the same sort of way. I feel that the teaching of science should be exhilarating. Astronomy, for instance, is a very suitable subject—I mean quite elementary observational astronomy—even without any great precision; the students should have their attention drawn to the heavens, and told what can be said about them by those who are competent to tell it. They must be told; they cannot find out things for themselves, and it is our business to pass on our accumulated knowledge to the next generation. They will absorb it fast enough. And if they can go further, and get an idea of law and order, some notion of Kepler's laws in dealing with the planets, and begin to realise gradually the way in which Newton's theory explains and vivifies these laws, turning them from empirical pieces of observation into a unity, like beads threaded on a string, having the thread of law and order running through them all—if the whole course could lead up to a full appreciation of what Newton had done in astronomy—whether it is feasible for the majority of men I do not know—that alone would give a feeling of reverence for science which could not but bear fruit in the future.

*Experimental and Descriptive Physics.** *By* PROF. R. A. GREGORY, *Chairman of the British Association Committee on Science Teaching in Secondary Schools.*

When science teaching was introduced into Rugby School about 60 years ago the instruction consisted of lectures and demonstrations. The apparatus used at that time was elaborate and expensive, and it was not adapted for use in experiments by individual pupils. It was the late Prof. A. M. Worthington who first showed that courses of practical work with simple appliances could be carried out successfully in school physical laboratories. His "Physical Laboratory Practice," published in 1886, embodies the experimental course developed by him at Clifton College, and afterwards introduced into many other schools.

The next stage in the teaching of physics in Secondary Schools is represented by the syllabus of Physics and Chemistry put forward in 1895 by a Committee of the Incorporated Association of Headmasters. In this syllabus measurement was insisted upon at the outset as the chief factor in scientific work and as the basis of scientific reasoning. It was urged that a large proportion of time given to science in schools should be occupied by the pupils in performing actual experiments themselves, and that the object should be to impart, not merely information, but chiefly the knowledge of method, and, with this object in view, that the instruction should be given in strictly logical order. The course was intended to be truly educational, and not for the preliminary training of chemists or physicists alone; it was devised to provide clear ideas of scientific method and reasoning rather than to cover a wide range of information, such as may be gained from text-books or other descriptive literature.

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* Communicated before the meeting.

The adoption of the principles of this syllabus by the Oxford and Cambridge Local Examining authorities, and by other bodies concerned with school examinations, has determined the character of school instruction in physics during the last 20 years. Questions have been set which demand personal acquaintance with experimental work and results rather than knowledge gained by reading or from lectures, with the consequence that most of the time available for science teaching has been taken up with laboratory exercises, and descriptive lessons have occupied a secondary place in the course. As precise laboratory work requires much more time for its profitable performance than is needed by didactic instruction, the actual ground covered must be less, even though the mental training is infinitely better. So it has come about that the majority of pupils in our Secondary Schools never get beyond the stage of determining specific and latent heats in their course of physics, and they leave school knowing nothing of the principles of such everyday instruments and appliances as field glasses, electric bells, telephones, periscopes, spectroscopes, dynamos, motors, incandescent and arc lamps, fuses, and many other applications of physical science met with in everyday life.

This condition of things has for some time been felt to be unsatisfactory, and much attention is devoted to it in the Report of Sir J. J. Thomson's Committee on The Position of Natural Science in the Educational System of Great Britain. It is pointed out that a general course in science, suitable for all pupils, should fulfil two functions—namely: (*a*) it should train the mind of the student to reason about things he has observed for himself, and develop his powers of weighing and interpreting evidence; (*b*) it should make him acquainted with the broad outlines of great scientific principles, with the way in which these principles are exemplified in familiar phenomena, and with their applications to the service of man. In recent years the first of these functions has been over-emphasised and the second neglected. The teaching of physics has become almost entirely quantitative, and little has been done to create and foster interest in the broad field in which every intelligent mind can find pleasure. It is important to provide training in scientific method, but it is even more important that every pupil should be given a sense of the wonder and value of scientific achievement. All science is not measurement and all measurement is not educational, nor does it appeal to all pupils, whereas few are incapable of appreciating the intellectual and industrial value of discoveries in the realm of physical science.

It ought to be recognised that much of the instruction in elementary practical measurements of length, area, volume, mass and density should be given by the mathematical staff, and not by the science staff.

This was recommended by a joint committee of the Mathematical Association and the Association of Public School Science Masters in 1910. Such work can be done in an ordinary classroom with the simplest apparatus, and is thus more easily co-ordinated with the mathematical lessons than when carried on in a room specially devoted to it. The course of measurements, including the use of simple balances, need very seldom exceed 20 hours of practical work ; and there can be no doubt that it is of the highest value in giving actuality to the mathematical teaching. Unfortunately, mathematical teachers have usually little sympathy with these practical methods of instruction, so that the fundamental measurements which give reality to their formulæ and operations have to be done in the two or three hours a week devoted to science. If teachers of physics were relieved of this preliminary work, it would be possible to cover much more ground than can be touched under existing conditions.

Both Sir J. J. Thomson's Committee and the British Association Committee on Science Teaching in Secondary Schools, which issued its report last year, urge the necessity of directing more attention to those aspects of science which bear directly on the objects and experience of everyday life.

At present, as instruction in science proceeds in the school, there is a tendency for it to become detached from the facts and affairs of life, by which alone stimulus and interest can be secured. It is important that every opportunity should be taken to counteract this tendency by descriptive lessons in which everyday phenomena are explained and the utility of discovery and invention is illustrated.

The summary of the section of the British Association report dealing with principles and methods of science teaching states the main points to be borne in mind, whether in the teaching of physics or any other branch of science. It is as follows :—

" The observational work by which the study of science should begin opens the eyes of the pupils, and may be used to train them in the correct expression of thought and accurate description. The practical measurements in the classroom have for their object the fixing of ideas met with in the mathematical teaching. Every pupil should undergo a course of training in experimental scientific inquiry as a part of his general education up to a certain stage, after which the laboratory work may become specialised and be used to supply facts which may be a basis for more advanced work or to prepare pupils for scientific or industrial careers.

" At suitable stages, when pupils are capable of taking intelligent interest in the knowledge presented, there should be courses of descriptive lessons and reading broad enough to appeal to all minds and to give a general view of natural facts and principles not limited to the range of any laboratory course or detailed lecture instruction, and differing from them by being extensive instead of intensive.

" Finally, the aims of the teaching of science may be stated to be (1) to train the powers of accurate observation of natural facts and phenomena and of clear description of what is observed ; (2) to impart a knowledge of the method of experimental inquiry which distinguishes modern science from the philosophy of earlier times, and by which advance is secured ; (3) to provide a broad basis of fact as to man's environment and his relation to it; (4) to give an acquaintance with scientific words and ideas now common in progressive life and thought."

The most urgent need at the present time is to devise suitable courses in which these principles are followed, so that all pupils may be given a broad view of the results of scientific work, as well as be introduced to experimental methods of inquiry. Under existing conditions, which are largely controlled by prescribed syllabuses and external examinations, there is little opportunity for teachers to direct attention to the useful applications of science on one hand, or on the other to awaken interest in inspiring principles. If, however, an authoritative body like the Physical Society can produce a practicable scheme of work suitable for all pupils up to about 16½ years of age—that is, to the stage of the First School Certificate—and will secure the adoption of the scheme by the examining bodies which will award this certificate, a valuable service will be rendered to teachers and to education.

Prof. GREGORY said: It is many years ago since I was actually engaged in the teaching of physics, and in those days it was true of me—what naturally would be true of a teacher of physics, or a teacher of any other subject—that my interests were largely concentrated upon my own particular branch of science. Well, for a quarter of a century now I have been actively associated with science in a periodical form—in "Nature"—and this has enlarged my views and taken me outside the interests alone of the physical laboratory, or of physics as usually understood, opening up a larger field than was at all possible when I was engaged in teaching one branch of science only. It is, therefore, with a certain amount of diffidence that I appear this afternoon before teachers to express any opinion upon the teaching of physics; indeed, I do not propose to trespass upon their field by giving expression to any decided opinions of my own as to details in connection with the teaching of physics; but I may be permitted to refer to one or two views which are now becoming general as to the importance of science teaching as a whole and of physics in particular.

In all subjects one really recognises three stages of development. There is, first of all, the stage when the advocate and the reformer come forward with views as to the necessity or the advisability of introducing what may be a new science, or a new spiritual doctrine, or a new system of ethics. At any rate, you have that definite stage of the fervid reformer who proposes to rejuvenate the world with his new teaching. That is the most active stage, and in the case of physics the science passed through it about 60 years ago, in the time of Kingsley and Huxley and Tyndall, and the carrying through of the first course of physical science teaching at Rugby School. The next stage is when teachers begin to reduce the work to formulæ, to produce a logical course, a plan of teaching; in a word, to create a system out of those views which had been advocated and put forward in an ungeneralised form. You have the same thing in the case of a new gospel. After the early stages of its apostles, you developed a formula and a liturgy. Next you reach the third stage when you begin to inquire into the foundations of your belief and teaching. These three stages are definitely represented in physics. In my early days, as a student, physics was made very interesting The experiments were qualitative, but they covered a wide field. As a boy I was introduced to fundamental measurements, heat, light and electricity, very superficially it is true; but one did see how wonderful these things were—quite as wonderful to many minds as the world of natural history, to which Sir Oliver Lodge has referred. About 20 years ago the Association of Headmasters produced a scheme of work in which the dominant principle was quantitative measurement and discovery by the pupil at all stages. The main part of the teaching was to be quantitative instead of qualitative. That scheme was after a time adopted by the University Locals authorities, and, as those who are concerned with secondary schools know, this meant that it became the usual course in secondary schools—at any rate, in schools of the State-aided type. You had the stage of precision, a definite course being given, and the pupil being expected to follow on from one point to another. Well, the result is, as those of you who are familiar with State-aided secondary schools know, that the majority of the pupils leave these schools, never having got beyond the determination of specific heat and latent heat. They could not mend an electric bell or adjust a fuse; they know nothing of the common things around them—apparatus and instruments; they never get near them, because precise measurements occupy all their time.

Well, we have had the same kind of thing in the teaching of other subjects. Algebra passed through the same stages. In my early days we were never expected or given the opportunity of using a table of logarithms until we had passed the theory of indices. In the case of geometry, we studied practical construction as a thing apart from the ordinary course of theoretical studies. Modern languages have been through much the same stages of development. From the grammatical stage we passed to the direct method of teaching languages, and now modern language masters are beginning to consider whether there should be a combination of the direct and the grammatical method, or which is the better of the two. We have these stages in the teaching and growth of most subjects. I am reminded of an essay written by a boy on the three ages of man. "The first age of man is when you are young and think of all the wicked things you are going to do when you grow up; that is the age of innocence. The next age is when you have grown up and done all the wicked things you thought about; that is the prime of life. The third age is when you are old and repentant, and that is dotage." Well, I think we have perhaps arrived in the third stage—at least in my own case—for I have been responsible for a fair number of books in connection with the teaching of physics in which the logical method had to be followed. I am repentant in a way, because I am very much in favour of what Sir Oliver Lodge has said—I have spoken and written on that point many times—with regard to the distinction between the training of the boy who is going to become a serious student of physics at the University perhaps, and is entering engineering or industrial work later, and the place of physics in the general education of those who are not taking up the subject in connection with a career. There can be no question that if you are given boys and girls up to the age of about 16 years, when one may suppose that the general education will terminate, the present course of physics as followed in State-aided secondary schools, devoting a year or more to fundamental measurements and a year or more to heat, and reaching optics and electromagnetism towards the end, perhaps, is most unsatisfactory. The difficulty comes in when arranging a course which will satisfy both your aims. I think it can only be solved by associating with your laboratory course—which is essential in order to get an appreciation of scientific method—descriptive lessons, which must not be limited to or follow the work done in the laboratory, but be independent. Usually, as teachers here very well know, the demands of examining authorities are such that it is necessary to adapt the descriptive lessons closely to what is done in the laboratory. The questions that are set demand intimate acquaintance with laboratory details, and it is necessary to elaborate these in the lessons upon physics connected with the course. Now I should like to see particularly a course that was of a descriptive and inspiring kind, such as that to which Sir Oliver Lodge referred, and it is for teachers to consider, in association with such a Society as this, whether it is not possible to produce such a combination suitable for the instruction of every boy or girl attending a secondary school. The ignorance of common apparatus and common phenomena now exhibited by the man in the street—I will not say anything about the woman in the street—is such that I am sure it ought to be deplored by a progressive nation. A friend of mine told me that a few days ago, in a very important camp, the colonel placed an ordinary recording barometer in the messroom. An officer came in and looked at the instrument, and after examining it for a time he said: "I don't see what good this thing is. It only shows 30, and this room's as hot as blazes!" He would have benefited by a little introduction to a descriptive course of physics in schools. Another officer, a little more familiar with the subject, looked at it and asked its object. "Yes," he said, "that is all very well; but what is the good of a barometer which doesn't show dry or rainy or fair weather?" Apparently he thought a barometer of no use at all unless it was one of the sort which you can tap and watch the pointer move. That is the kind of thing we have in consequence of the neglect of the general study of science in our schools. The general body of pupils is neglected in comparison with the small proportion of pupils who are specialising in the subject. All I wish to plead for this afternoon—all that is embodied in my remarks on this matter—is that we should try to produce some combination of descriptive teaching with the laboratory course, to appeal particularly to the general student, instead of to the specialist only.

The Place of Physics in the General Education of Boys up to the Age of Sixteen.

Mr. C. L **Bryant**, M.A., of Harrow School, said: Mr. President, ladies and gentlemen, I think it will be wise of me to confine my remarks almost entirely to the public schools, for about them my knowledge is most certain. During the past four years, as one of the Secretaries of the Association of Public School Science Masters, I have had occasion to learn fairly intimately the conditions prevailing in the public schools. Also I shall restrict myself to speaking of the place of physics in public school education.

There is quite an extraordinary unanimity of opinion among teachers that boys ought not to specialise before the age of sixteen. The First School Examinations now are meant to test this general education. What can be done between the ages of sixteen and eighteen (when many boys pass on to the universities) in the matter of limiting specialisation for those who wish to continue the study of science, is a different problem altogether. The schoolmaster who thinks at all—and most of us do, in spite of the prevalent opinion to the contrary—is always asking himself why he should teach the subject on which he is chiefly employed; and the aim of the science master—I hope I may be forgiven the statement—is not so much to teach science as to produce good men of sound judgment. The kind of judgment fostered by scientific training has this against it, that it must almost certainly be slow. The judgment of any special case of science requires special consideration from all the circumstances affecting it, and the process takes time. The same kind of judgment is required in a player of football, but in his case a very short time limit is imposed. But the great importance of science in education lies in the field in which evidence must be sought. It is not in tradition or in history or in any written book of the past that the seeker must look, but in the open book of Nature, of which, so to speak, there is a daily publication. And then, of course, there is the obvious utility of the subject matter. In order to describe our general aims, I cannot do better than quote from a memorandum drawn up by the Association to which reference has been made. This memorandum dates from October, 1916, and refers to the place of science in general education. It states that the first aim of science is the search for truth, and it goes on to say that its subsequent establishment depends upon accurate observation with constant recourse to Nature for confirmation. The one aim of natural science is, in effect, the search for truth based on evidence rather than on authority. Honest study of the subject implies accurate observation and description, and fosters a love of truth. The special value of natural science in the training of the mind and character lies in the fact that the history of the subject is a plain record of the search for truth for its own sake. Secondly, with regard to utility, there are certain facts and ideas in the world of natural science with which it is essential that every educated man should be familiar. A knowledge of these facts assists men (*a*) to understand how the forces of Nature may be employed for the benefit of mankind, (*b*) to appreciate the consequence of cause and effect in governing their own lives, (*c*) to see things as they really are, and not to distort them into what they wish them to be.

Following that definition of aims, there was drawn up a scheme entitled "Science for All,"* designed to indicate the kind of science teaching suitable in the general education of all boys. This has the universally expressed approval of the Association. It ranges over a wide field, from the universe to the electron. In it a deliberate effort has been made to wander from one main road of science to another. Thus, sound and the physiology of the ear; light and that of the eye; biology and the work of Darwin and Pasteur; discovery and the lives of the discoverers; and—in my own application of it—creation and the

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* Copies of this may be obtained at 6*d.* each from Mr. C. L. Bryant "Ludlow," Harrow.

Creator are not divorced. So it seems against the spirit of the thing to pick out the parts which relate to physics, though I have done so as this is the subject in which you are most interested. But first let me give you the main outline of the scheme.

(1) *Main Headings.*—The Universe. Solar System. Earth. Igneous and Sedimentary Rocks. Volcanoes. Glaciers. Fossils. Coal.

Atmosphere. Life of a plant. Fermentation. Pasteur. Animal Kingdom. Balance of Nature. Darwin. Simple Agriculture. Simple physiology and hygiene.

Natural resources of the Empire.

Mass. Weight. Density. Falling bodies (Galileo and Aristotle). Force and Work. Liquid and gaseous pressure. Diffusion. Capillarity and surface tension. Applications of the above.

Study of Atmosphere. Combustion. Respiration. Water. Limestone, sandstone, clay. Conservation of Mass. Laws of Combination introducing Chemical Theory. Flame. Hydrc-carbons. Coal-gas. Nitrogen, sulphur, chlorine and their simple compounds. Acids, bases and salts. Properties and extraction of metals. Alloys. Iron and steel. Petroleum. Coal-tar products. Oils, fats, soap, glycerine. Sugar.

Sources, effects and transference of Heat. Thermometers. Investigation of Heat quantity. Heat and temperature; thermometric scales. Calorimetry. Change of state; vapour pressure. Heat values of fuels. Heat and Work. Horse-power. Mechanical equivalent. Engines.

Rectilinear propagation of Light. Photometry. Reflection and refraction. Mirrors and lenses. The eye. Telescopes and Microscopes. Dispersion.

Wave motion. The ear Pitch, loudness, quality. The gramophone.

Magnets. Lines of force. Terrestrial Magnetism. Cells. Electromagnets. Telegraphs. Conductors and Insulators. Electroscope. Potential, E.M.F. Effects of current. Resistance. Ohm's Law. Current Induction. Microphone and Telephone. Dynamo and Motor. Electrical energy. Lamps. Heat and Work. Units.

Now follow details of the physics included in the scheme. These are very similar to the details of the physics which have been taught up to the present in a great number of schools. If one likes, one can subdivide them into heat, light, sound, and magnetism and electricity. But if you look at the text for some of the details you will see how in almost every case there must be some reason in the everyday experience of the pupil for treating of the subject.

(2) *Details of the Physics Included in the Scheme.*

MECHANICS AND HYDROSTATICS.

Falling Bodies. Galileo's disproof of Aristotle. Acceleration Conservation of Mass and Momentum. Energy.

Based on a machine; in the same way other courses can be based on other machines, *e.g.*, internal-combustion engine, etc.

Hydraulic press. Transmission of pressure: fluids: Solids. Force. Spring balance. Pressure and Tension. Weight. Mass. Work. Pulleys.

Hydraulic Machinery (Lifts, Jacks, etc.).

Pressure and depths in liquids. Lock gates. Density. Specific Gravity (by direct weighing).

Air pressure. Vacuum Brakes. Barometers. Density of Air. Height of Atmosphere. Distinction between Gases and Liquids (briefly). Boyle's Law. Aneroids. Barometer movements and weather.

Diffusion of Liquids and Gases.

Flotation (Ships and Balloons). Hydrometers.

Lactometers. Density of Sea-water. Deep-sea sounding apparatus. Divers. Archimedes' Principle and applications.

Pumps. Lift pump and force pump. Syphons. Air pump. Water-wheels (overshot and undershot). Pelton Wheel. Steam Pressure. Force on piston of steam engine. Simple manometer. Pressure Gauges. Turbines (impact and reaction). Water-rams. Capillarity. Wetting.

HEAT.

A. Heat quantity as experimental investigation.

B. Effects of heat on the sizes of solids, liquids and gases (qualitatively). Sources of heat. Conduction of heat. Davy lamp. Convection of heat. Ventilation and heating of houses. Thermostats. Winds. Ocean currents. Water-tube boilers. Radiation (briefly) Expansion of solids on heating. Annealing. Rupert's drops. Unequal expansion of different metals; platinum and glass. Pendulum Control of clocks and watches.

The fixed points of the thermometer. Temperature scales. Air-thermometer. Absolute zero. Clinical, maximum and minimum thermometers. Effects of salt and pressure on melting-point. Freezing mixture. Regelation. Effect of salt and pressure on boiling point. Marcet boiler.

Temperature and heat. Units of heat. Specific and latent heat.

Change of state. Evaporation. Condensation. Distillation (whiskey, petrol). Cold on evaporation (sparklets). Heat on compression (air compressors, Diesel engine, refrigerators). Vapour pressure. Rain. Dew. Hoar-frost. Snow. Hail. Relative humidity. Wet and dry bulb thermometer.

Change of volume on melting. Bursting of pipes. Freezing of ponds. Change of volume on boiling; hence steam pressure.

Radiation. Characteristics. Cold nights with cloudless sky. Thermos flask. Glass fire-screens. Hot-houses.

Heat values of fuels. Heat and work. Forms of energy. Work and power. Horse-power. B.H.P. of an engine. Mechanical equivalent. Conservation of energy. Steam engine (general outline). Energy losses. Work done by expanding steam in cylinder. I.H.P. Efficiency.

LIGHT.

Rectilinear propagation of light. Candle-power. Intensity of illumination. Photometers. Plane mirrors Laws of reflection. Searchlights. Images in spherical mirrors (geomet. construction ; no formulæ) Refraction. Index of Refraction. Apparent depth of pond. Internal reflection.

Prisms. Lenses. The eye. Accommodation. Defects of vision. Spectacles. Magnifying glass. Astronomical telescope. Terrestrial telescope. Magnifying power. Field of view. Microscope.

Dispersion. Colour. Rainbow.

SOUND.

Nature of waves on water surface. Frequency. Wave-length. Nature of sound waves in air. The ear. Vocal cords. Pitch due to frequency. Doppler effect. Siren. Velocity of sound in air. Wavelengths. Gramophone. Claxon horn.

MAGNETISM AND ELECTRICITY.

Magnets. Magnetic fields Polarity. Earth's Magnetism. Compass needle. Cells and batteries. Electromagnet. Telegraphs. Conductors and insulators.

Electrification. Electroscope. Potential. Capacity Condensers. E.M.F. of cell shown on electroscope.

Effects of current ; heating, chemical and magnetic. Measurement of current by any of these effects. B.O.T. unit of current. Current capacity of wires. Ammeters. Resistance. Ohm's Law. Voltmeters. B.O.T. unit.

Current induction. Microphone. Telephone.

E.M.F. in wire moved across magnetic field. Dynamo. Magneto. Alternating current. Commutator. Motor.

Self-induction. Induction coil. X-rays. Wireless telegraphy.

Electrical energy. Heat. Electrical power. Watt. B.O.T. unit of energy.

Lamps. Parallel circuits Wiring of houses.

Now for one or two explanatory remarks on it. This is meant to be taught to all boys between the ages of fourteen and sixteen, and the majority of them will not be interested specially in science later on in life. The courses are therefore designed to be self-contained, and are not formed so as to lead on to the more specialised study of science. Why should the grammar of science, and that only, be imposed on all ? The drudgery of learning the elements of heat, of chemistry, and so on, is exactly comparable with the study of the Latin Primer, which is of small value except to those who are to make a special study of the Classics. This desire for the exact knowledge of detail is characteristic of adults, and, even so, is only indulged when there is reason for its satisfaction. How many of us would care to learn the Russian alphabet unless we had reason to know that we should be able to utilise our knowledge of the language ? And why impose such tasks on the defenceless boy ? The boy does not wish at first to learn about abstract principles of science. Again, speaking from my own experience, as I got on in years I did find the fascination of probing a subject to its very depths ; but that was not in my boyhood's time, and, unless I mis understood Sir Oliver Lodge just now, it seemed to me that his experience is different from mine. It is a serious matter to differ from Sir Oliver Lodge on any point of science or education, and especially on a point which combines the two, but in my judgment, and from my own experience, I should say that boys learn very quickly from the concrete, whereas older people are more willing to learn from elementary principles. I remember some years ago an academic discussion on the value of following out the line of historical discovery in science in teaching any special subject, and there were some who thought that one ought to take principle after principle as it was discovered, and so work up to the common ground of knowledge of the present day; others held the contrary view. But I recollect a remark by the Headmaster of Oundle, who said that he believed in the historical treatment of the subject, but that history ought to be read backwards!

Imagination comes before discovery in natural science; the vision comes before the detailed work. It is this vision and the incentive to work that we wish to give the boys first of all. Too many people credit boys with the desire for learning for its own sake. That is about equally rare among boys and among grown up people. Each beginner in science should learn something of man's place in the universe.

To my mind, the most glaring fault of the science teaching of public schools at the present day is the neglect of mechanics, a subject most important for the study of nearly all branches of science.

To pass from our aims and ideals to the conditions under which we work, examinations hamper us far too often. Examiners are often without experience of the teaching in the schools. They seem unable to break away from the antiquated rudiments of chemistry, heat and optics. No teaching is more hampered from these circumstances than is science. The subject matter of the scheme I have outlined is admittedly difficult to examine in, and there must be more co-operation between examiner and teacher. Again, I strongly oppose the grouping of science with mathematics in the First School Examination. Training in science is at least as important as the study of language, which has a group to itself; and there are other reasons of school politics which make the grouping undesirable.

Boys should be graded for science separately from literary subjects A boy should not go up through various science divisions of a school according to the form he happens to be in, if the placing of the form depends chiefly upon literary subjects.

The system of scholarships on entrance to public schools has the result that the boys, having begun to specialise in Classics, naturally stick to the subjects they have found to "pay." Boys come from the preparatory schools to the public schools rather below the age of fourteen and leave at the age of eighteen or nineteen. In the majority of State aided secondary schools, on the other hand, training is continuous from ten to seventeen. In the preparatory schools it is rare to find science taught at all. This is not altogether the fault of the preparatory school masters, for they teach what the public schools require in the matter of entrance examinations; but the result is that the sons of the wealthy begin their training in science about four years later than their less fortunate fellows! Even in these days there are public schools where boys—usually the most able—can pass through without having had any training at all in science. We do insist that the boys should be regularly instructed in preparatory schools in Nature study and Practical Measurements. By the former, habits of observation and description can be cultivated, while the latter form the basis of all work in physics.

Physics for Older Boys.

Dr. T. J. Baker, King Edward's High School, Birmingham, said that the honorary secretary had very kindly asked him to offer some remarks on a part of the subject which was obviously closely related to that dealt with by Mr. Bryant. The course of science work through which the boy had passed before he reached the age of 16 years might influence the rate of his subsequent progress very considerably. The boy who had assimilated the "Science for All" course should be in a favourable position to profit from the more exact treatment which became necessary when he began to specialise. The question whether boys should leave school between the ages of 16 and 17 in order to enter directly upon the more specialised science work at a university could have only one answer from teachers who had had experience in schools. A boy of 16 or 17 must possess most unusual qualities of concentration and determination if he was to profit from the instruction offered him. The discipline of the school kept the boy definitely at work for prescribed periods; at school he could not "cut" the lesson. But at the university he was a freer agent, and, within limits, he might or might not do the work. Boys of that age had not sufficient stability to be trusted to work steadily under such conditions. Quite apart from this consideration, it should be borne in mind that a university was not making the best use of its powers when it undertook science teaching which could be carried out efficiently by the schools. One had heard of professors who had stated that they preferred their students to come to them with no knowledge of their particular subject rather than that they should have started it at school. But the experience of such professors must have been particularly unfortunate. Sir Clifford Allbutt, the Regius Professor of Physic at Cambridge, said in 1915, with regard to the fairly recent regulation which allowed boys to take part of the first M.B. direct from school, that the results had improved largely on those obtained in the past. The men coming forward were not merely as good as in the past, but were far better. The assumption had been made frequently in discussions that 16 years represented the age at which boys had reached the standard of general education corresponding to matriculation. Statistics showed that the number was greater for matriculation at 17 than at any other age. The speaker had found from the records of the past 10 years in his own school that the average age of the boys who had matriculated was about 17 years and two months at the time when they began their specialised course in science. This seemed to be in fairly close agreement with the London University statistics, and showed that it was unsafe to assume that post-matriculation work began on the average at 16 years. The report of Sir J. J. Thomson's Committee recommended a lowering of the age for open scholarships at the older universities from 19 to 18, and that an appreciable fraction of the school time of prospective candidates should be devoted to literary work. Under such an arrangement the university course in science would of necessity start from a lower level than at present. It must be remembered that, after all, only a certain small percentage of boys were likely to proceed to the older universities, and therefore it was desirable to consider how the others would be affected by a scheme which insisted upon a less specialised post matriculation science course.

The boy who left school immediately after matriculating and proceeded to one of the newer universities would start at once on a completely specialised course in medicine, engineering or pure science. He would not be required to work at any literary subjects. If he remained at school the new scheme would insist upon part of his time being given to literary work. Parents would be likely to consider such time as wasted, and boys would be removed from school when they had matriculated, and would be sent to the university at once. There they would be doing the work, and that work only, which would lead to a degree. It was open to grave doubt whether the newer universities would alter their first year courses to bring them into line with the post-matriculation courses proposed for schools, and nothing short of this would be effective in preventing boys from evading the proposed less specialised post-matriculation courses in schools, and thus missing the advantages which such courses offered from the point of view of general education.

The Board of Education, in proposing the institution of two examinations in grant-earning schools, plainly recognised that many boys would be more than 16 years of age by the time they were ready for the first examination. The second examination was designed for those who had continued their studies for two years after the stage of the first examination. The secretary had told him (the speaker) that it would be useful to state his views on the maximum in physics which could be expected from boys of 18 who had been specialising for two years in science, and he was in some difficulty in this respect, because in his experience very few boys had been specialising in science for two years at the age of 18; indeed, most of them would have been specialising for not much more than one school year. Among 12 boys who were taking the Intermediate Science and the first Medical Examination, the average age was 17 years and seven months (a younger group than had been usual for some years), and eight of these boys had been specialising for one school year only. Taking these circumstances into consideration, it appeared that the standard of work required by an Intermediate Examination in Science represented something like the maximum which might be expected at an age approximating to 18 years. It was also necessary to remember that the majority of these boys would not be candidates for open scholarships at Oxford or Cambridge, and that an estimate of what might be expected from them should not be based upon the performances of the few who were likely to be successful in such competitions. He thought that in some of the post-matriculation syllabuses brought forward there were certain requirements which might with advantage be left to a later stage, such as shear strain, viscosity, surface tension, Carnot's principle and its applications, fourth power law, diffraction and polarisation of light; although these might be dealt with successfully in the second year when the boys would probably be rather over 18 years of age. The difficulty of dealing with these boys was often increased by their imperfect knowledge of the principles of mechanics. In a fair number of cases they had taken the subject for matriculation, but it frequently happened that one came across boys who had never studied it at all seriously before entering upon the Intermediate Science course. This arose from the circumstance that the parent often postponed his decision as to his son's future career until he had matriculated. Referring to a two years' post-matriculation syllabus put forward by one school, he said that in his opinion it was altogether too ambitious, for it aimed at preparing boys of 18 for a B.Sc. examination with honours. The schemes proposed by two other schools corresponded fairly closely with the usual requirements of an Intermediate Examination in science. The speaker distributed among those present cyclostyled copies of a syllabus of physics for boys of 18, which dealt with the phases of the subject treated under the headings of heat, light, sound, magnetism and electricity, and included some suggestions relating to the additional work which might be required from boys of about 18 years and six months, who had given nearly two years to the subject. He concluded by saying that he had no wish to dogmatise with regard to exactly what should and what should not be done, but he put forward this syllabus as a suggestion.

Syllabus of Physics for Boys of 18.

HEAT —Temperature. Expansion and consequences. Quantity of heat. Specific heat. Conduction. Change of state. Latent heat. Vapour pressure. Hygrometry. Mechanical equivalent of heat. Elementary radiation.

LIGHT.—Rectilinear propagation and consequences. Photometry. Reflection at plane and spherical surfaces. Refraction and dispersion. Prisms. Use of spectrometer. Lenses. Simple optical instruments. Defects of vision. The spectrum. Colour, Velocity of light.

SOUND.—Nature of waves. Loudness, pitch, quality. Velocity of sound in air and other media. Vibration of wires, transverse and longitudinal. Determination of frequency. Resonance. Vibration of air in pipes. Beats. Doppler's principle.

MAGNETISM.—Induction. Inverse square law and consequences. Magnetic field Pole strength. Moment of a magnet. Magnetisation of iron Earth magnetism.

ELECTRICITY.—Simple electrostatics. Charge. Potential. Capacity. Energy of charge. Specific inductive capacity. Cells. Accumulator. Properties of the current. Measurement of current. Force on current circuits in magnetic field. E.M.F. Resistance. Ohm's Law and consequences. Ampere, volt, ohm. Heat production in circuits. Joule's Law. Electric energy. Electric power. Electrolysis and electro-chemical equivalents. Electromagnetic induction. Transformer. Induction coil. Principle of dynamo and motor. Thermo-couple and its uses. Microphone. Telephone. Potentiometer.

Additions for Boys of about 18 *years* 6 *months who have given nearly Two Years to the Subject.*

General Physics.—Motion in a circle. S.H.M. Simple treatment of moment of inertia. Compound pendulum. Density of earth. Young's modulus. Simple experiments on torsion. Surface tension. Deviations from Boyle's Law.

Heat.—Enough elementary thermodynamics to give ideas about adiabatic processes and efficiency of heat engines. Indicator diagram. Radiation.

Light—Simple application of wave theory to reflection, refraction, interference and diffraction. Simple polarisation. Application of Doppler's principle. Colour.

Magnetism and Electricity.—Permeability. Susceptibility. Intensity. B. Calculation of field in simple cases. Inductance. Ideas about electric oscillations leading to wireless telegraphy. Discharge of electricity through gases

The President said that Mr. Sanderson, the headmaster of Oundle School, who was to have taken part in the discussion, had sent a telegram regretting his inability to be present. He (Mr. Sanderson) had added that he would like to see a small committee formed to report on science work in schools of different types and to consider science work in advanced courses and continuation courses. The President added that Mr. Sanderson had lost his son in one of the recent advances, and that he had our sincere sympathy. He thought Mr. Sanderson occupied a prominent place amongst those who believed it was best to teach boys by going from the concrete back to principles ; or, to use the illustration which had been brought forward in the course of the discussion, to argue back from the steam engine to Boyle's Law. Or, as Mr. Bryant had expressed it, he believed in teaching history backwards.

*The Teaching of Physics at Osborne and Dartmouth.** *By* C. E. Ashford, *M.V.O., Headmaster of the Royal Naval College, Dartmouth.*

Osborne takes in the same material as the Public Schools. There is no competitive intellectual test at entry. But the final aims differ to some extent : Public Schools aim at developing a typical character, but foster all possible varieties of mind; while the executive branch of the Navy should contain men who conform to one fundamental type of mind as well as of character. It may be assumed that any one officer will carry out the same duties as any other at some time in his career, though not in the same order, the chief exception being that the abler half will also specialise for 10 or 15 years. So the education must be framed with regard to these typical qualities. For example, Cadets should develop into men of action rather than students, ready to continue indefinitely to learn for themselves by observation and experiment rather than from teachers or books. They must be many sided, keen on everything, from machinery to strategy ; they must possess or cultivate the power to make instant decisions, and an aptitude for improvising means to meet novel conditions; they must have their knowledge at call, and not merely know where it is to be found; they must face facts and abhor prejudices. Such an analysis suggests that experimental science should be an admirable discipline for them.

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* Communicated before the meeting.

Cadets enter Osborne at 13–8, and normally end their general education on leaving Dartmouth about 17. After this they take a technical course of eight months in special cruisers, and then go to sea as officers. The course in Physics begins with an unsystematic treatment of Hydrostatics, to introduce Cadets to an orderly habit of thought, observation and description. They also begin at once in the Laboratories and Engineering Workshops.

The effect of the rapid alternation of theory and actual practice throughout the course is found to be extremely favourable to both, not least in the case of young boys. Workshop practice makes demands on the recently acquired store of scientific knowledge in a fashion well suited to the budding Naval Officer, who will have to think while standing up and doing a job; a limited amount of knowledge ready at his finger tips will be of more service to him than a much wider amount available only in a special environment. And the habit of learning from what he is actually doing will stand him in good stead in carrying on his own education. Conversely, the concrete experience of the shops stimulates his interest in theoretical work, and supplies the teacher with endless illustrations.

During following terms the subjects taken are Statics, Heat, Magnetism and Direct-current Electricity, Kinetics and Hydrostatics. The time devoted to them is roughly three or four hours a week in lectures, one hour for preparation and two in the laboratory. The average Cadet does not, as a matter of fact, get as broad or as full a knowledge of pure Physics as a boy who has passed up the modern side of a good public school—*e.g.*, Geometrical and Physical Optics and Acoustics are not touched—and there is more Applied than Pure Physics in those subjects that are dealt with. But in a public school the weaker boy never reaches the upper levels, while practically every Cadet who enters Osborne takes the Passing Out Examination, and is put through the full course; this produces considerable difficulties in organisation, but that is perhaps a smaller evil than persisting in a course for all which is adapted only to a few, and letting the remainder leave it uncompleted. The practical nature of this compound of pure and applied Physics with Engineering seems to appeal better to the average boyish intelligence than does pure Physics alone, and there are fewer weaklings than might have been expected. The abler Cadets of each batch are given work that is more difficult and more generalised and abstract, as a foundation for the specialist courses which they will take later, and various methods are adopted to stimulate them to exert their full powers.

Many problems have presented themselves; one is the question of text-books. There is a strong temptation, especially where much ground has to be covered in a short time, to give full lecture notes or a text-book written for the prescribed course. These give no practice in extracting items of knowledge from a mass of irrelevant material; it became clear that we must sacrifice speed of progress for ultimate gain, and it was found that the early retardation was almost balanced by more rapid progress at the end even of this short course.

The relation of Mechanics to Physics is another difficulty. In many Public Schools this is treated as a branch of Mathematics; it comes late in that course, and many science masters have made a successful effort to teach the earlier stages of Physics without depending on it. There is much to be said for this procedure, if only because comprehension of mechanical principles seems to be a function of age, and the commencement of Physics cannot wait for the attainment of that age and the subsequent courses in Mechanics. At Osborne and Dartmouth Cadets are concerned from the first in their engineering with mechanical ideas and principles, so that to obtain co-ordination it is necessary to start serious work in Statics as part of the Science course earlier than would seem advisable on other grounds, and to introduce Kinetics as soon as Cadets are at all ready for it. Experience seems to show that these young boys can learn the mechanics required, but with considerable difficulty and much more slowly than if they were older. The method of teaching must be adapted to their special needs; but probably such a change would be good even for older boys with a better mathematical equipment who are liable to trust to their power of dealing with symbols and avoid the trouble of thinking mechanically. The system adopted is not to start with metaphysics and abstractions, but first to take four or five machines (*e.g.*, screw-jack, Weston block, &c.), and examine fully in the laboratory their behaviour, measuring and plotting their velocity ratio, efficiency, &c., under various conditions. Moments arise naturally, and their forces are treated (in the class-room) as vectors, with the ordinary logical development of statics. At the age of about 16, kinetics can profitably be dealt with; it is treated experimentally so far as possible, with kinematics only as it arises and mathematical formulæ postponed; prominence is given to momentum, work and energy, and problems solved by their use whenever convenient.

Several methods of teaching the earlier stages of Electricity have been tried; that which has proved most satisfactory is a term's work of a qualitative and unsystematic kind, dealing with the behaviour of such things as electric bells, dynamos, motors, induction coils, primary cells, resistance coils, etc. Then follows the electrolytic measurement of current; next the electrostatic voltmeter, using voltaic and not frictional electricity. With the ability to measure Current and P.D., Ohm's Law can be verified, and the conception of Resistance deduced. It has been found that this process results in an absence of the confusion between Current and P.D. which is common among beginners.

It may be worth adding that Osborne and Dartmouth must, equally with Public Schools, provide a liberal education. Probably in neither case is this perfectly balanced, the schools laying undue stress on languages and the Naval Colleges on the mathematical and scientific branches; but inasmuch as the Humanities take an important place in the work, our experience in teaching Physics is probably of more value to those concerned with Public Schools than with Technical Colleges.

Mr. C. E. Ashford, M.V.O., said: It may be worth while pointing out how far the conditions of the teaching with which I have been chiefly concerned in recent years differ from what you have already been considering in public schools and State-aided secondary schools, in order that anything I give as the results of the experiments going on at Osborne and Dartmouth in the teaching of science may be more intelligible. In the first place, the goal of the teaching is clearly defined in its earliest stages, which is not the case in almost any other scholastic institution. Speaking negatively, the aim most emphatically is not to get certain boys with credit or otherwise through an examination. The Admiralty provides the training for its cadets, it does its own examination, and it utilises the services of these boys for the rest of their lives. Therefore we work under an almost ideal state of things. There is no inducement to design or modify the training so as to suit a few selected individuals as an advertisement, while overlooking the claims of the remainder

In the second place—and in this it is comparable with what should, perhaps, be the aim of a great many schools—the desire is to turn out men of action—practical men, not students primarily. After all, the object of being a student is usually that you may later on profit by the knowledge you have acquired and your practice in acquiring it. What I mean by men of action rather than students is men who will continue all through their lives to learn by doing things for themselves, by observing and experimenting, rather than by studying under teachers or in books. Naval officers must be many-sided, keen on everything, from machinery to strategy, and must possess or cultivate the power for instant decisions and for improvising means to meet novel conditions. They must have their knowledge handy; they must not merely know where facts are to be found. Some of these boys are to be specialists in various branches, but all of them will pass their lives as men of action. Moreover, they will be men who do not do things themselves, but make others do them. That may seem a subtle distinction, but it is much harder to direct the uninitiated in undertaking a task than to do it oneself. You will realise from this what kind of scientific training it is necessary to give at the naval colleges.

For instance, the measurement of simple physical quantities, which appealed to the early educators in science for various reasons, does not appear to be an ideal introduction to laboratory work. The beginner is bored by measuring a number of specific gravities and densities and so on. Still more strongly do I believe that boys are not ready to verify general laws at the beginning of their career in physics. What they want, then—I am speaking of boys between the ages of 13½ and 15—is to inquire into the behaviour of machines and apparatus. When that is done the boy will naturally seek the why and wherefore—in other words, the theory; and I venture to say that not only the naval officer, but the ordinary man in the street, will carry on both the process of acquiring knowledge of physics and the desire for it by having to use new machines.

There is another specific difference between the naval training and the training given in the majority of ordinary public schools in respect to the inclusion of a parallel and contemporary course in engineering workshops. It is essential that the men who are to take charge of those "boxes of engines," as battleships have been called, should be engineers in their very bones, and accordingly they go through a course of engineering from the age of thirteen. The reaction of this practical course on their theoretical science work is most beneficial; it provides a boundless wealth of illustration for those who are responsible for teaching the theory, and those who are teaching in the workshop are continually making demands upon the theoretical knowledge of the boys. It would appear likely to make the physics taught a great deal more applied than pure; it does so in the early stages, and my own feeling is that this is psychologically correct. The concrete and the applied is the correct material with which to provide a boy up to about fifteen. I believe the boy will naturally come to a systematic course later on, because he feels ready for it. As one effect of an early course of engineering, I would instance the teaching of mechanics. A science master at a public school, faced with the difficulty of starting boys on physics at fourteen with a blank ignorance of mechanics, may well decide to turn the difficulty by designing an introductory course of physics which does not involve any mechanical knowledge. There is very much to be said for this procedure; but if you have an engineering course running side by side with the physics course, both beginning at just over thirteen, you must grasp your nettle and attempt to teach mechanics at that age. That can be done satisfactorily, though it is slower, and the boys find it harder. My own belief is that the power to assimilate the principles of statics and kinetics is a function of age; but if you set the boy to deal with full-sized machines, to study their behaviour by taking four or five and thoroughly examining their mode of action, their velocity ratio, their efficiency, &c., and plotting the results—such machines as the screw jack, the Weston block, and so on—he will get a great deal of advantage out of them. He can go on later to the standard systematic course of statics, and at the age of about sixteen he is competent to undertake the very much more difficult questions of kinetics. These, I think, should be handled in the same way—that is to say, experimentally—keeping work, energy and momentum paramount, and kinematics as little prominent as you can. Kinematics should be treated only as it arises. The early treatment of mechanics is one of the difficulties which an engineering course introduces, but I believe that the reality and the concrete nature of the work and the interest which the boys exhibit in it, would alone justify having a course of engineering running parallel with the other.

The third point in which the conditions at Osborne and Dartmouth differ from those in public schools is that the course for every boy who enters is the same. I would support what Mr. Simmons has said—that it is the "educated citizens" we have to provide for rather than the specialist. Specialists have themselves damaged the training of boys in science because they have designed courses for them suited to future specialists, and not in the least applicable to those ordinary boys who will go through life—and these form the vast majority—not as specialists, but nevertheless as men from whom very serious and important work is expected. In the Naval training colleges, looking at a given entry of a hundred boys, we realise that perhaps half these boys will not live to reach high command in the Navy, but of the other half 25 per cent. will have been specialists of no mean order, and even upon the remaining 25 per cent. the fate of England may hang. I believe that the individual genius will not suffer at all from a course that is psychologically correct for the average boy.

Dr. Allen told me that it might be of some interest if I gave a very brief sketch of the methods by which we introduce the subject of electricity. What we have found to be most satisfactory is an unsystematic term's work at the age of about 14½, when both lecture and laboratory hours are devoted to a study of the behaviour of a jumble of the electrical apparatus and machinery which can safely be put into the hands of boys. This includes glow lamps, galvanometers of a simple electromagnetic kind, simple telegraphs, electric bells and motors, both toy and commercial sizes. There is no question of measurement whatever; they have not reached that stage. Let them deal also with dynamos, cells and electro-plating; this leads to the quantitative course by providing a measure of current. The introduction of potential difference is an obvious difficulty, but it can be overcome with surprising ease by taking sufficient thought.

I think I can give an outline of the process we adopt in a few words. We show a gold-leaf electroscope with the two leaves insulated from one another, and show the attraction between the gold leaves when a P.D. of about 100 volts is maintained between them. From that to an ordinary electrostatic voltmeter, as an indicator of a P.D., is a very short step. We proceed to define difference of potential by saying that a battery of *n* cells in series, each of *V* volts, gives a P.D. of *nV* volts; by building up a battery—which, of course, the boy has done before in his qualitative way—of a series of cells of known E.M.F. the voltmeter can be graduated. The boy thus acquires a qualitative and quantitative idea of a P.D. which is free from any danger of confusion with a current. Having, then, a graduated electrostatic voltmeter, and an ammeter graduated by electrolysis, Ohm's law can be verified. Then we introduce resistance as a ratio; and once you have got Ohm's law and your three practical units clearly in mind you can proceed extremely rapidly to quantitative questions arising from motors and dynamos. It is in such work, I believe, that boys are stimulated to learn the science of electricity, and where even your future specialist is going to profit. He gets at the age of sixteen a sense of reality which he does not get from the proof in the laboratory of various principles which are too abstract to interest him. A really rapid advance like that carries the boys at a run through and over various difficulties. A sixteen year old cadet deals unhesitatingly with fairly large motor-generators and their switchboards, making all connections to suit specified conditions, and he finds it logical and desirable to study such things as magnetisation curves and absolute units; they follow more naturally in that process than in the other. I do not for a moment wish to seem to belittle the systematic and logical course—it is vital and essential; but I hold that it is usually begun too soon.

I would mention one further point. I was under the impression that this might be an over emphasis on applied physics, tempting to the teacher because it interested the pupil without making demands on his mathematical powers, and so on; but I got one piece of evidence which rather went to show that this is not so. I calculated the Bravais-Pearson coefficient of correlation between science and mathematics in a considerable number of groups who pass out from Dartmouth at the age of 17, and I took the same co-efficients of correlation between science and engineering. It is curious that science and mathematics have a co-efficient of correlation of 0·81 and science and engineering of 0·78. There is, therefore, a tendency for science as taught even on these lines to be more closely correlated with mathematics than with engineering, so I am inclined to think that such an experiment as this, though carried out under special conditions, may be of interest to those chiefly concerned with theoretically pure physics.

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*Physics in State-aided Secondary Schools.** *By* A. T. Simmons, *B.Sc., A.R.C.Sc.*

I gladly avail myself of the opportunity of directing the attention of the Physical Society to some aspects of this wide subject which have impressed me in my work of inspecting the teaching of Science in Secondary Schools, the larger number of which, it should be said, are Schools receiving financial aid from the Board of Education.

In the schools to which reference is here made the age of nearly all the pupils ranges between nine and seventeen years. In most of the boys' schools and some of the girls' schools the attention of the pupils from twelve to sixteen years of age is given to the study of experimental Physics and Chemistry, and in most of the schools Physics and Chemistry are both studied in each of the four years under consideration. The average amount of time devoted to Science may be fairly put at four hours a week throughout the course. It is often less than this in the first year and more during the last year of the course. The general rule is to divide the time equally between Physics and Chemistry, though in some schools one subject only is studied during the last year, as external examinations are being prepared for. The present discussion is concerned only with the instruction in Physics.

The most common practice still is to devote the first year to preliminary measurements, such as those of length, area, volume and mass. More enlightened headmasters are, it is true, beginning to call much of this work Practical Mathematics, and to transfer it to the Mathematical course, and to make the Mathematical staff responsible for it; but, so far as my experience goes, this wise plan is even yet the exception rather than the rule. The second and third years are most commonly given to easy experimental work in hydrostatics, statics, dynamics and heat. In some schools a few experiments in light may be introduced in these years, or perhaps a few in electricity; but these cases may be regarded as exceptional. Pupils in the fourth year are commonly preparing for external examinations: hitherto it has been for a Matriculation Examination, and in the future it will be for the First General School Examination imposed by the Board of Education. The authorities—generally the Universities—who conduct these examinations have hitherto divided Physics up into a number of subjects, the divisions adopted being different in the case of different Universities, and Candidates generally offer one division of Physics only for examination purposes. In the case of most candidates these divisions are either (*a*) Heat, Light and Sound; or (*b*) Electricity and Magnetism.

Now most of the pupils leave school when this examination of matriculation standard has been passed, though a select few stay on for a further one or two years' study, and then have the opportunity of studying those branches of Physics to which as yet they have received no introduction.

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* Communicated before the meeting.

It is thus seen what the average boy or girl leaving these secondary schools has done in Physics. If, as is often the case, he leaves school before taking the fourth year of the Science course outlined above, or even if he first takes the General School Examination and offers Heat, Light and Sound, he passes out into the world completely ignorant of Electrical Science. He has studied Physics at School, forsooth, and has not even the most elementary knowledge of the manifold applications of electricity in human affairs.

Even if he has offered Electricity and Magnetism in his General School Examination, as schemes are at present arranged he has few opportunities of gaining an elementary acquaintance with the applications of Electrical Science to the industrial arts. Whatever he may or may not know of Electricity and Magnetism, it is certain either that his knowledge of Heat and other engines, &c., and of the phenomena of Light and Sound, is of the scantiest or his ignorance of them is complete.

The fact seems to be that hitherto school courses of Physics have been designed by men who have taken it for granted that all such work done in school is merely preliminary to a fuller course which will be entered upon after school days are over; that all school pupils who offer Science at the Matriculation Examination will continue to study the subject through the Intermediate and Degree stages of the University. But a very small fraction only of the pupils proceed to colleges of University rank and continue the study of Science in this way; and the teaching of Physics in schools will not be satisfactory from the point of view of the average boy or girl until it is recognised that the school course must be complete in itself, and that the knowledge the pupils gain there represents for the large majority all they will ever learn of Physics or any other branch of Science.

The task before the Physical Society is therefore clear. It should decide: what parts of Physics is it of imperative importance that every educated citizen should know, and what steps should be taken to make it easy for the secondary school pupil to acquire this knowledge at school?

I suggest that one of the first improvements for which the Physical Society should work is the substitution of a more general course of Physics—including the essentials of its various branches—for the more detailed and quantitative study of a few constituent parts of Physics. It must be understood, however, that this substitution will never be effected until the University authorities are willing to accept for their Matriculation Examinations such a general course of work as I have adumbrated in place of their present more specialised syllabuses in separate branches.

Another matter of almost equal importance is so to arrange the young pupil's introduction to the study of Physics that full use is made of the enthusiastic eagerness with which he first enters the physical laboratory. The boy of twelve of whom I am speaking comes generally from a course of Nature Study, where, if he has been well taught, he has handled and observed plants and animals. He may also have had some manual work and handled tools. He comes to the Physics Master enamoured with the idea of doing things for himself. He has learnt, moreover, to compare his work in Nature Study, and his experience of tools, with the formal work of the mathematical classroom, and Mathematics has no doubt suffered from the comparison. And what is provided in Physics in most schools to feed this young enthusiasm? The youngster is set to measure lengths in inches and centimetres, to find a value for π, to measure areas by counting squares, and so on. Instead of handling simple apparatus and *doing interesting things*, he finds himself working over again in a slightly different way problems with which he is or has been concerned in the Mathematics classroom, and instead of a change of occupation he feels he has only secured a change of room. To obviate this disappointment, the practice might well be that exercises which require only paper and simple mathematical instruments, and can be done equally well in an ordinary classroom, should be transferred from the Science Course to the scheme of Mathematics, and that the work of the Physical Laboratory should begin where the use of water and the balance are necessary.

Care must be taken, too, to provide the young beginner with variety and interest. Measurement may be easily overdone in these early stages. Surely it is possible in the first year at least to have a minimum of such accurate measurement, and to introduce instead a succession of everyday topics like pumps, syphons, floating bodies, and so on, the principles of which could be studied from working models.

And in every year of the course the teacher must strive to maintain the interest and to introduce variety. Of course, there must be sufficient practice in measurement in later years; but primarily it should be remembered that the needs of the ordinary boy whose work in Physics must be completed at school have to be borne in mind, and that if he is to gain a general idea of this wide subject only a strictly limited time can be given to any one branch

This argument suggests that in the future more use must be made of experimental demonstrations by the teacher, and it would be of service to Science Masters if the Society could indicate what subjects should—from the point of view of the place of Physics in a general education—be selected for individual experiments by the pupils themselves, and those which should be reserved for demonstration by the Master.

Mr. A. T. SIMMONS, B.Sc., A.R.C.Sc., Inspector of Secondary Schools for the University of London, said: Mr. President, ladies and gentlemen, I propose to be as brief as I can. I have summarised my remarks in the printed abstract which has been distributed already. I shall now try to index them. I think most of the teachers present came imagining that they were going to get a great deal of help in teaching physics out of the Physical Society. I am anxious that they shall not be disappointed, so I propose to formulate very briefly two problems to be left with the Physical Society for their more leisurely consideration. My knowledge of the teaching of physics has been obtained chiefly by watching the teaching in schools, those aided by the Board of Education, for the most part. I think I may summarise very rapidly what goes on in these schools. The age, roughly, varies from nine to 17. Physics is taught between the ages of 12 and the leaving age. In the first year of the teaching of physics the pupils are subjected to the inevitable fundamental measurements; in the second and third years they are given a little statics, hydrostatics, heat, and sometimes—not very often—a little light; and, if you get a more original teacher, there may be just a suggestion of electricity. Then the fourth year arrives, and, it being a State-aided school, an examination begins to loom in the near distance, and the pupils cannot go on with their general course of physics because the examination is conducted, not directly by the Board of Education, but for the Board of Education by the universities, and the universities have not, it would seem, a very intimate knowledge of what goes on in the schools. The first examination is called the First School Examination, and it has to serve two purposes. It has to convince the Board of Education that the work in the schools which have been aided by them has been satisfactorily done, and, in the second place, it has to find out whether the work has been suitable to serve the purpose of matriculation—that is, as an entrance to a university course. Now notice the condition of things carefully. The university does the examining, and in nearly every case it prescribes the syllabus, and yet it knows apparently very little about the work in the schools; but it does know what future work is required of the student who enters one of the colleges of the university. So the university authorities lay it down that for matriculation the pupil must offer either Sound, Light and Heat, or Electricity and Magnetism. And what is the result? The result is specialisation either in Sound, Light and Heat or in Electricity and Magnetism. Mechanics is, it is true, sometimes taken, but never really as a branch of physics, only as a branch of mathematics under the direction of the mathematics master who performs no experiments. These boys, after they have got through the examination, leave school at about 16½ or 17. Now, surely here is something for the Physical Society to contemplate. The pupils' knowledge of physics, if they have taken Sound, Light and Heat, has included no mention of Electricity and Magnetism. They cannot put in a fuse, they cannot mend an electric bell, they do not know anything about wireless telegraphy or electric traction. On the other hand, if they have taken Electricity and Magnetism instead of Sound, Light and Heat, they know nothing about Sound, and very little about Light. Ought it not to be possible —is it beyond the powers of the Physical Society—to suggest a general course of physics and to indicate what are the fundamental facts and principles that the ordinray intelligent well-educated person ought to know, and could not they persuade the universities to let teachers submit a syllabus of a general course of Physics, and allow the matriculation examination to be passed in that?

I ask you now, in the second place, to bear in mind that the children who come to these schools usually arrive at the age of nine. They begin the study of science as soon as they arrive. My experience is that they take up Nature-study in a thoroughly practical way, handling plants and animals, drawing them, and really get to know something of the subject; and, besides this excellent work, the boys go—a great many girls, too—into the workshop and learn something of tools; and there they hear that when they get to the age of 12 they will have the privilege of going into the physical laboratory and starting the study of physics. They *really* think it is going to be a great privilege, and in due course they get there, and are, as I have said, put on to those inevitable fundamental physical measurements. They start measuring lengths in inches and centimetres, measuring areas by squared paper and so on; and they soon come to see that they have been humbugged; they find that they have done the work before, only it was then called arithmetic! For in good mathematical teaching the pupils are introduced to measures of mass, length, area and volume in quite a practical way, and they are put to mark time for a whole year! They say, "We have not changed the subject, we have changed the room." They came to the Physics teacher absolutely brimful of a desire to learn physics, and you have done your best to choke them off.

I have a rough rule that I apply as to how the study of physics should begin. I always say, "All the practical work you can do in the ordinary class-room you can safely call mathematics; but when it is necessary to use water and a balance and there is a chance of making a mess, it is a case for the physical laboratory." Put the boys on to use working models of pumps, syphons, simple machines almost at once, and you will arouse an immediate interest which it should not be difficult to foster and maintain.

Prof. F. **Womack**, Bedford College, said: At this late hour I shall only keep you a very few minutes. My outlook on the subject is necessarily a good deal narrower than that of several of the speakers who have preceded me. The students with whom I have to deal are of two classes—men students going in for medicine, and women students, who, in the main, are going to take up the teaching of science. As regards the former, at the present time they enter at about the age of 17½ to 18, the age of entry being just now a little younger than it used to be. I have for some years past taken statistics of the knowledge of physics possessed by these students, and I find that during the last four or five years no less than 34 per cent. have had no previous training whatever in the subject. They were brought up in school entirely on the classical or the modern side, and are destitude of physical and frequently of chemical knowledge. The choice of a career in the case of these lads has often been left to the last moment. The parents have not decided what they should do with the boys, or, perhaps, they have decided at the very last moment that the medical profession is the only thing they can put them into. No doubt, it is still the case in a large number of secondary schools that there is a great predominance of the teaching of classics, and the boy who tries to go over to the science side is at once stigmatised as of a lower intellectual type Of the remainder, a few have done a little heat and a little geometrical optics. They have worked evidently from a narrow university syllabus, and with hardly any exception they have had no mechanical foundation except those dry-as-dust linear and areal measurements. They have had nothing to do with measurements of acceleration, with the measurements of the ballistic pendulum, or friction. I feel strongly that the teaching of mechanics in the schools should be in the hands of physics masters. It should not be handled by mathematical masters. In some schools the difficulty is that the master has to teach both subjects. I was very glad to hear Sir Oliver Lodge's suggestion that the teaching of science should be humanistic. Only a few months ago, speaking to the Association of Headmistresses, I made the same suggestion. It would always be possible to make the teaching of physics and mechanics in a large town biographical and humanistic. One could show students in London where Davy, and Faraday, and Boyle worked, where Hawksbee lived, and so on. With regard to the teaching of girls, still fewer of them have had any prevous training in mechanics; the teaching of science is delegated as a rule, to the assistant mistress, who is expected to be cognisant of physics and mathematics and Scripture, and not infrequently of botany as well! It would be a gain if the student of mechanics and physics at school could be freed from any syllabus whatever. I realise that there is always pressure being brought to bear upon masters and mistresses to allow youths to go up for examination, but if the examinations could be rendered as far as possible free from any association with the cut-and-dried syllabus of university courses, it would be enormously to the gain of the students themselves, and they would come up to the colleges with their interest maintained.

A letter of apology for inability to attend and take part in the discussion as promised was read from Mr. G. F. Daniell, of the Education Department, London County Council; and the Paper by Prof. T. P. Nunn, of the London Day Training College, was taken as read.

*The Training of Teachers of Physics.** *By Prof.* T. PERCY NUNN, *London Day Training College.*

It is hardly possible to deal usefully with this problem without indicating one's position with regard to a prior question: What is the aim of Physics teaching in schools? Let me say, then, that it does not seem to me satisfactory to define that aim either as the communication of a certain body of knowledge or as a certain type of "mental training." Physics is more than the sum total of known physical truths and physical methods. It is a historic way of intellectual life, a living and growing tradition of interest and inquiry which has for centuries been one of the main currents in the stream of Western progress, and is one of the chief constituents of present-day intellectual activity. Our object in teaching physics as a branch of general education should be to initiate the pupil into that way of life, to put him to school to that great tradition; in short, to make him feel what it is to be, so to speak, inside the skin of the physicist, sharing his interests, ideals and outlook on the world, learning in a simple way to use his tools, and tasting something of his sense of joyous intellectual adventure.

The first requirement in the training of the teacher is, I suggest, that he shall have learnt Physics in this sense himself. It is by no means always satisfied. Even graduates who have taken respectable places in the Honours lists too often leave the university with a narrow, pedantic and lifeless knowledge of their subject which is an inadequate basis for school teaching. Their knowledge is carried as an external burden, instead of being a genuine organ of their intellectual life. They may know physics, but they have not *lived* it. It is beyond my province to inquire into the origins of this serious defect. I venture however, to think that it will be remedied only as the aims of school teaching and university teaching become better defined and more clearly delimited; the former becoming less academic and more broadly "human," the latter losing the character of a training for success in a competitive examination.

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* Communicated before the meeting.

In the second place, it is necessary for the teacher to have not only a technical but also a critical or reflective knowledge of his subject. The attitude of the young graduate towards the concepts and hypotheses of Physics is, as a rule, naive; he takes them at their face value, never having had occasion to consider how much they contain of convention, symbolism and metaphor. For the future engineer such considerations may be superfluous; his use of Ohm's law, for instance, will not be prejudiced even if he should not only speak but think of electricity as "juice." But the case of the teacher is different. His business will be to use scientific ideas in building up his pupils' minds; he needs, therefore, to understand the true nature of the materials with which he is to work. Moreover, scepticism of the instruments of scientific progress (I adapt a phrase borrowed from Mr. H. G. Wells) is not merely a tiresome habit of impertinent metaphysicians; it is a vital part of the business of science, though not one in which every man of science need be interested. The teacher's preparation should include, therefore, a critical study of the history and nature of Physics in the spirit exemplified in Mach's famous work on mechanics and his less well-known book on heat—that is, he should study the psychological situations in which the chief notions of Physics arose, and should trace the main steps in their development, his constant object being to determine the significance of the historical facts as stages in man's perennial struggle to extend his intellectual control of physical phenomena. Incidentally, he should gain from such inquiries a sound elementary philosophy of science, many valuable hints for teaching methods, and an intimacy with the heroes of Physics which will often enable him to give a desirable human touch to his lessons.

Next comes the professional training in the narrower sense of the term. This will, of course, include an apprenticeship to the practical arts of exposition, class management and laboratory management served under competent supervision, as well as opportunities for observing accomplished craftsmen at work. But a liberal course in methodology is, to say the least, equally important. By "methodology" is meant a discussion of the distinctive aims of science teaching and of its place in education, the proper scope and character of the curriculum in differing circumstances, the principles that should determine the order and methods of treatment at different ages of the pupil, and a comparative study of the best methods of teaching the more important items in the Physics course. To set down this bald catalogue is to give a very inadequate idea of the value of a well-conceived course in methodology. It is impossible to expand the items here, but I may refer to Section IV. of the recent B.A. Report on Science Teaching for a fuller indication of the contents of one important corner of the field covered by the term.

The practical questions where and how the professional training should be given have occasioned much difference of opinion. The Board of Education recognise two rather sharply contrasted arrangements: The training may be given either entirely in a secondary school or in a training college. In the second case it includes teaching practice under supervision in a secondary school. It is difficult to see how either plan can be really satisfactory. The first may promise "quick returns" in superficial efficiency, but cannot guarantee, as a rule, the liberal grounding in the theory of his profession which every teacher ought to have. The second plan suffers from the opposite defect; it does not, as a rule, secure that the student's theoretical studies shall be adequately backed with practical work or pursued in a sufficiently practical atmosphere. Sir J. J. Thomson's Report advocates a middle course; it proposes that the student shall spend one term in a training college or university training department, and two terms as a student-teacher in a secondary school. This compromise, which appears to have been recommended largely in view of the needs of the residential public schools, has its own weaknesses, of which the most obvious is a failure to provide for effective correlation between the theoretical work of the training college and the practical work of the school. Its strength is that it recognises the necessity of combining the two factors which the Board's schemes divorce. Training colleges have as yet secured only a precarious reputation in the secondary school world, but the Report is undoubtedly right in the view that they have a function that cannot generally be exercised in a secondary school. They should be to the teaching profession what the Staff College is to the Army: centres where the experience of the profession finds its clearest and most conscious expression, and is formulated for general use, laboratories of pedagogical research and clearing houses of educational ideas, foci of educational thought and enthusiasm.

It is worth considering whether the analogy of the Staff College might not well be followed in recruiting the *personnel* of the training college; whether, that is, some of its teachers should not be schoolmasters or mistresses whose qualifications include aptitude for reflection upon professional principles as well as notable practical skill and experience, who would be invited to withdraw for a period from the busy routine of school life and take a share in forming the next generation of their colleagues. But whatever may be the value of this particular suggestion, I am convinced that the way of wisdom with regard to training colleges is not to suppress or to ignore them, but to take serious pains to strengthen them for the better performance of their indispensable duties.

Mr. J. Nicol, B.Sc., Northern Polytechnic, Holloway, said: The course of science followed at the average secondary school has been very ably described and criticised by Mr. Simmons in his contribution to this discussion. The main fault that he has to find is that too little ground is covered, so that the boy leaves school entirely unacquainted with certain branches of the subject. This, I consider, is due to the object which has been aimed at in shaping the science courses. Professor Gregory has quoted from the report of the Thomson Committee two of the advantages to be gained from an education in science—(*a*) training in observation and reasoning, (*b*) the acquisition of directly useful knowledge; but he omits to mention the advantage which the Committee put first—namely, that "it can arouse and satisfy the element of wonder in our natures." We may say that science teaching of the best type will satisfy, and thereby stimulate the curiosity of the boy. It will in many cases provide him with a hobby for his spare time, and should frequently create an interest in scientific progress, the pursuit of which will serve as an intellectual recreation throughout life. Unfortunately, the first of these aims (training in reasoning and observation) has been allowed to overshadow the others in the preparation of schemes of science teaching for schools As a result, the greater portion of the time is spent in the laboratory, and the exercises there are arranged to form a progressive course, and parts of the subject not reached by the course are comparatively neglected.

Take, for instance, heat as it is taught in the majority of the schools. The measurement of temperature, of coefficients of expansion of liquids and gases, and of specific and latent heats is well done, because it is easy to arrange a series of experiments that the boys can do individually. In addition, these experiments yield definite numerical results, which enable the master to check the accuracy of the work and assess its value in marks without much trouble. Other parts of the subject, such as the relation of change of state to pressure, the transfer of heat, and the conversion of heat into work, which are quite as important and have quite as many applications to everyday life, are comparatively speaking neglected. Even when the mechanical equivalent of heat is dealt with the treatment is usually very academic, and the steam engine is hardly mentioned.

In light, again, the laws of reflection and refraction, and the formation of images by lenses and mirrors, are subjects that lend themselves to individual experiments with quantitative results. On the other hand, the subject of colour and the effects of mixing pigments and lights of different colours, with their important applications to colour printing, are hardly dealt with at all.

In my opinion the cure for this fault is to be found in making a much wider separation than is customary between the laboratory work which the boys do themselves and the lessons at which experiments are shown by the teacher. The time given to the former should (supposing no extra time is allowed for science) be considerably cut down. That this can be done with advantage in the ordinary school, where more than half the total available time is now spent in the laboratory, becomes obvious, when we remember that Mr. Ashford has told us that at Osborne and Dartmouth, where the instruction is intended to be of a particularly practical type, only one-third of the time given to physics is spent in the laboratory. The time thereby saved should be used for the purpose of short lectures, illustrated by what may be termed "pretty" experiments—that is, by experiments that are qualitative rather than quantitative in character. In addition, there should be lantern slides illustrating the applications of physics to everyday life and to industry. Everything should be done to make these lectures interesting, and the boys should be encouraged to read as widely as possible in connection with them. Books approximating to the science reader rather than to the matriculation text-book type should be supplied, and several books should be read through rather than one exhaustively studied. The treatment of the subject in the lessons should be inductive rather than deductive, so that the boys' general knowledge is first increased with examples that satisfy their curiosity, and then the general principles underlying the applications can be brought out. This is preferable to the present practice of attempting, first, firmly to fix the underlying principles, only introducing the applications as illustrations of these if time permits ; or, if time does not permit, leaving them out altogether, in the belief that when, later on in life, the boy becomes acquainted with the applications he will remember his principles well enough to apply them.

If the physics work is so divided it may with great advantage be co-ordinated with the rest of the school work, so that the laboratory course gives practice in mathematics, while the lecture course affords material for English composition. This latter point is of great importance, for it may be said that the treatment outlined and advocated above is that which was in use twenty years ago in those schools which took any interest in science teaching. Then the science lesson was regarded by the majority of the boys simply as an opportunity for slacking, and the failure of this old teaching led to the introduction of the heuristic method, in which each boy was supposed to discover everything anew for himself under the more or less skilfully concealed guidance of the teacher. A strict adherence to the heuristic method is now generally admitted to be unsatisfactory, because of the inordinate amount of time it requires, and it is recognised that the boy may frequently gain far more advantage from watching a carefully devised and prepared demonstration carried out by the teacher than from doing a less satisfactory experiment himself, provided that his attention is really held, and that he afterwards describes what he has seen and learned.

It must, however, be emphasised that the choice of a method of teaching and of a particular course of work is of subsidiary importance if a really good teacher is obtained. From this point of view the present method of selection and treatment of teachers is open to criticism. Fortunately, there are very few cases where a teacher who does not know his subject is asked to teach physics ; but there is a tendency to go to the other extreme, and pay attention only to specialised academic qualifications in making a choice. At the present time, for instance, many secondary schools are trying to retain a proportion of their boys at school to do more specialised science work from the ages of 16 to 18. For this work it is considered necessary to have a teacher with a first class honours degree. In order to obtain this a man must have had for several years an extremely academic education, which does not fit him for work as a schoolmaster. Boys take a great interest in new applications of science to everyday life, such as steam engines, motor cars, aeroplanes, wireless telegraphy, X-rays and so on. In a secondary school there is no teacher of engineering, and the boy should naturally look to the physics master for help and suggestions for reading and experimenting in these subjects. For such work a less specialised training than that of the first class houours man in physics is more suitable.

The question of the training of the teacher has been specially discussed by Prof. Nunn ; but in this, as in the training of the boy, there is too great a tendency to regard the matter as finished when the master leaves the training college or the boy the school. In the latter case we must, before all, aim at creating an interest, so that the boy will continue to learn from books and everyday life. In the case of the master, we must realise that physics is a living subject, and is at present moving forward as fast, or faster, than any other branch of knowledge. The teacher will, therefore, need to keep himself up to date, and at present he seldom receives sufficient encouragement to do this. Annual subscriptions to learned societies, payments of fees for advanced courses, the expense of attending, say, lectures at the Royal Institution or a British Association meeting, and the cost of books are in many cases beyond the means of science teachers, especially of those with a family. It would be in the interest of the school if the master were assisted financially in keeping himself up to date. A mere increase of salary does not do this, as it still leaves the choice of spending the money in other ways, and in these days of salary scales for everyone the return to the teacher on money spent by him in keeping thoroughly efficient is somewhat uncertain.

To sum up, the teaching of physics stands in need of broadening and revivifying. This can best be done by endeavouring to keep it in touch with everyday life, and so to defeat any attempt to standardise it and use it merely for the purpose of giving a training in logical method in the way that geometry was used in the days when it appeared in the school time-table as Euclid.

A Physics Laboratory for a Secondary School.

Mr. E. Smith, B.Sc., Leyton Secondary School, said : Mr. President, I should like to present the subject from the viewpoint of a master actually engaged in teaching in a secondary school, where all the pupils have to go through a course of physics from eleven years of age to sixteen

Efficient teaching presupposes, firstly, the existence of a proper scheme ; and, secondly, facilities for working it successfully. The provision of a proper scheme is generally easy. The authorities, the headmaster and the teacher are all in sympathy, and the capable teacher is given a free hand. The facilities may be classed under two heads : (i.) Laboratory ; (ii.) teaching facilities. These last two matters, which include questions of finance, organisation of a curriculum, the actual teaching time of the master and the time available for correction, preparation and repairs are apt to cause divergent views amongst governors, headmasters and teachers.

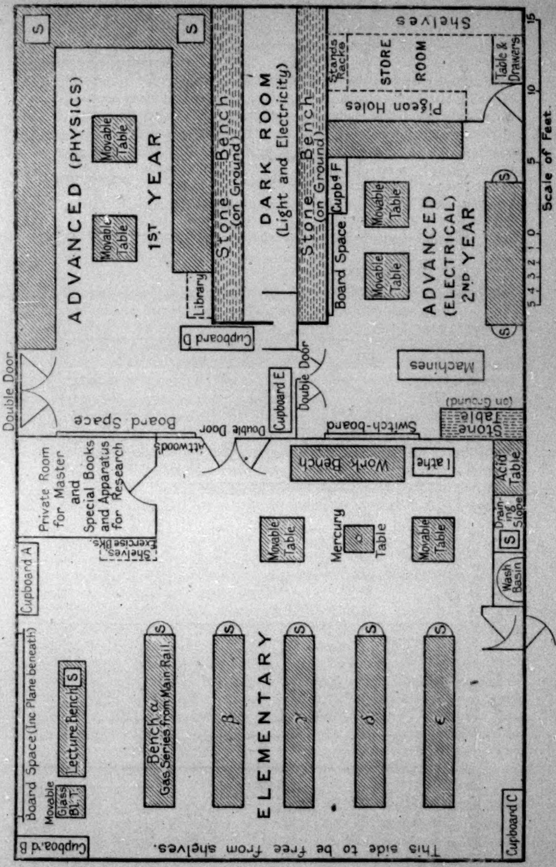
With regard to laboratory facilities, the course of teaching for younger boys and girls must be mainly qualitative, for one of the most important ends in elementary teaching is to make the young student form accurate mental pictures.

I show on the lantern screen (Fig. 1) the design of a laboratory measuring 60 ft. by 36 ft., and divided into two sections—namely, elementary, for students up to 16 years of age ; advanced, for students from 16 to 18 years. The course has in the last year of the advanced section an electrical bias, which has been rendered necessary, I believe, by modern developments.

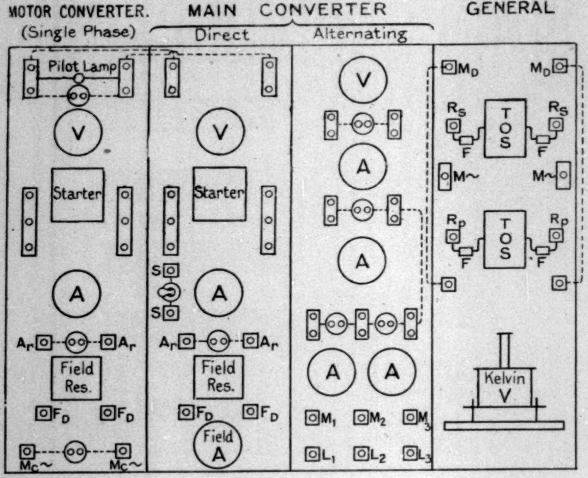
The arrangement of the elementary section should be such as to provide for rapid alternations from practical to lecture work, and the

PHYSICS.

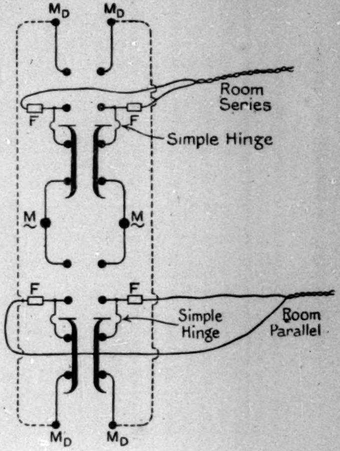
Suggested Laboratory for Complete Course.
(36×60 ft.)


ADVANCED (PHYSICS)
1ST YEAR
Movable Table
Movable Table
S
S
Stone Bench (on Ground)
DARK ROOM (Light and Electricity)
Stone Bench (on Ground)
Stands Racks
STORE ROOM
Shelves
Pigeon Holes
Table & Drawers
Library
Cupboard D
Board Space
Cupbd F
Movable Table
Movable Table
ADVANCED (ELECTRICAL) 2ND YEAR
S
S
Double Door
Double Door
Double Door
Cupboard E
Machines
Switch-board
Stone Table (on Ground)
Board Space
Cupboard A
Private Room for Master and Special Books and Apparatus for Research
Shelves Exercise Bks.
Attwoods
Work Bench
Lathe
Acid Table
Movable Table
Mercury Table
Movable Table
Drain-ing Slope
S
Wash Basin
Board Space (Inc Plane beneath)
Lecture Bench S
Movable Glass Bl. T.
Bench α Gas Series from Main Rail S
β S
ELEMENTARY
γ S
δ S
ε S
Cupboard C
Cupboa.d B
This side to be free from shelves.
5 4 3 2 1 0 5 10 15
Scale of Feet.

SWITCHBOARD.


MOTOR CONVERTER.
(Single Phase)
MAIN CONVERTER
Direct
Alternating
GENERAL
Pilot Lamp
V
Starter
A
A_r
Field Res.
F_D
$M_C\sim$
S
Field A
M_1
M_2
M_3
L_1
L_2
L_3
M_D
R_S
TOS
F
$M\sim$
R_P
Kelvin V

Throw-Over Knife-Contact Switch.


M_D
F
Room Series
Simple Hinge
$M\sim$
Room Parallel

benches are arranged to suit that purpose. Stools are provided in the gangways, yet the students can all face the teacher at any moment. If you take a simple experiment, in frictional electricity for example, it will not take more than five or ten minutes to perform, yet you will require to consider the results in detail and connect them, if possible, with every-day phenomena

A large amount of free space has been allowed for experiments of a bulky character, such as for wave motion and certain sections of mechanics, and one wall has been left free for other mechanical experiments.

The advanced laboratories have provision for more continuous work. Each laboratory, elementary or advanced, should have its own electrical circuit, with a "series" connection, if possible, to a direct-current town supply; with suitable single-pole terminals on each bench coming from a bank of lamps which supplies the whole room. This arrangement does away with the wasteful and untidy use of primary and secondary cells. Secondary schools have fairly long holidays, and secondary cells are liable to deteriorate.

A dark room is essential for optical and galvanometer work. To avoid the consequences of interruption, it is better to have a blackened entrance stopping direct rays of light than to have a doorway.

The advanced electrical laboratory should be fitted with "series" and also with "parallel" circuits that can be instantly changed over from direct to alternating sources when required. The switchboard (Fig. 2) should be governed by a foolproof switch, and all the connections should be made on the surface by the students. The governing switch avoids the necessity for constant and close supervision by the master. In an electrical laboratory there is much scope for the creation of apparatus for abolishing bad contacts and the provision of subsidiary sources of E.M.F., and also many opportunities for using commercial apparatus without impairing the educational efficiency of the course.

The machines advised for a flexible installation with a direct-current supply are:—

i. A single-phase rotary converter that may be coupled to

ii. A rotary converter having six slip rings; and

iii. A small three-phase induction motor with a wound rotor having external connections to a resistance.

The designs given show many smaller details. They allow for a provision of fundamental necessities, and yet permit the physics master full scope for original adaptations in his course.

With regard to the provision of teaching facilities, there is one point I should like to raise. In the science teaching in the State-aided school the amount of time allowed to the master for preparation, correction, epairs, etc., is, I think, generally inadequate. I should estimate that in an advanced section half of the total time would be certainly necessary, whilst in the elementary section one quarter would suffice. This point requires much consideration at the present time.

Mr. W. R. Cooper (communicated) was glad to see that there was a consensus of opinion in favour of making physics interesting. This, he thought, was most important If it were not too unorthodox, he would have liked to have seen the meeting addressed also by some representative schoolboys. The trained physicist of middle age was apt to forget the boy's point of view. It would be well to draw up two courses of instruction, one being for boys of, say, 11—14 years, and the other for older boys. The object of the junior course would be to arouse interest, pure and simple, and to enlarge the scope of the mind; measurements and principles would be excluded. For the older boys measurements might be introduced. It was very important to give instruction to the younger boys because their minds were very receptive, but it was essential to interest them. Nothing less inspiring could be imagined than, say, the determination of specific gravity. The small boy could see no use in it; he probably became weary of what he considered useless and uninteresting, and preferred to dabble in chemistry, in which he could see something going on. Many preparatory schools were probably at fault in giving no instruction in physics.

Mr. F. B. STEAD, H.M. Inspector for the Board of Education, said: Mr. President, I was very glad to have the opportunity of being present at this meeting. If I venture on any observations I hope it will be distinctly understood that I am speaking for myself alone, and am not putting forward views officially endorsed by the Board of Education. The Board has not formulated any view as to the methods of science teaching in secondary schools, save that in 1909 some observations did appear in the annual report of the Board—a document, I am afraid, which is not very widely read. Among the many interesting Papers which were circulated I turned naturally first of all to the Paper by Mr. Simmons on "Physics in State-aided Secondary Schools," and I found myself in agreement, broadly speaking, with what he said as to the general scope and conditions of work in these schools. I would only plead that if there is a rather depressing uniformity such as he seemed to indicate in the work of these schools, that uniformity is not due to any efforts to secure uniformity made by the Board of Education. The uniformity, so far as it exists, is due to certain paramount conditions by which the teaching of all subjects, including science, is affected, namely, examinations, the general teaching tradition, the previous education of the teacher, and, last, but not least, the text-books, which tend naturally to affect the teaching in the school. With regard to the time to be given to science in secondary schools, Mr. Simmons says that the time devoted to science may be fairly put at four hours a week throughout the course. I think that is really a high estimate, certainly for girls' schools. If he means four periods, and not four hours, I rather agree. With regard to the preliminary work in mensuration I do not think it does absorb the quite inordinate amount of time which was at one period given to it. I quite agree that this work should be done by the mathematicians. The difficulty is to get the mathematicians to do it. I may say that I myself and my colleagues have been campaigning for the last 8 or 10 years at least—with a certain amount of success—in favour of limiting the amount of time given by the science teacher to that dull work of mensuration. This kind of work is, however, to be found not only in the State-aided schools, but also in one or two public schools of my acquaintance.

I should like to make one or two points respecting the teaching of physics, which have either not been dealt with by previous speakers, or which might be emphasised still more than they have been this afternoon. First of all, I would say that the physics teaching is very much influenced by the desire of the physics teachers to utilise the school laboratory at all costs. It results from this that the physics course consists too often of a series of practical exercises, mainly quantitative in character, carried out in the laboratory, each exercise designed to be completed within the limits of a single lesson period of an hour or an hour and a half. It follows from this that the laboratory exercise that can be done in one lesson period tends to become the unit of teaching. That is a fundamental error. The exercises in consequence are often disconnected from one another, too little attention is given to the actual results, not enough time is left for discussion, alternative methods of arriving at the result are ignored, and, last and worst of all, the relation of the particular exercise to the larger subject to which it belongs tends to be entirely forgotten. The boys often unconsciously pursue a course of practical physics following, perhaps, some text-book of practical physics and get very little out of it beyond a certain facility in manipulation and the power to carry out instructions. In the second place, measurement, in the larger sense, tends to be thought of as an end in itself. I should like to illustrate. The first introduction of a boy to the subject of specific heat is the practical expression, "find the specific heat of copper." Of course, it is not an *introduction* to the subject of specific heat. It is no doubt important that the boy should be able to find out the specific heat of copper, but it is much more important that he should realise that there is such a property as specific heat. If you ask the boy what is the effect of solar radiation on the surface of land when that surface consists of sand or clay or snow, he will often be very much puzzled to give you an answer, in spite of the fact that he has done some determinations of specific and latent heats, and carried out one or two experiments on conductivity. That, of course, is not right. Again, every schoolboy finds the index of refraction of glass by one of two methods. If you ask him to tell you anything about the index of refraction of diamond you are not likely to get much of an answer; and if you go on to ask him how it is connected with one of the most characteristic properties of the diamond, you get no answer at all. All this shows how wrong it is to make measurement an end in itself, and how extremely important it is that the course of practical physics should be helped out by adequate discussion and attention to results, and should not consist of disconnected exercises.

Attention has been drawn to two views this afternoon concerning methods of teaching physics. One is the view of Mr. Ashford and Mr Bryant that you should begin with the machine and work back to the principle; the other view favours the opposite method of treatment, indicated by Sir Oliver Lodge. I venture to make this suggestion, that the difference—apparently a very marked difference between these two views—is greater in appearance than in reality, and that differences in method of teaching which are startling enough when expressed on the platform tend to disappear when one is face to face with the problem in the laboratory or the lecture-room. I think we should find that Mr. Ashford and Mr. Bryant, if we met them at home in their own laboratories and classrooms, would be quite well aware of the importance of logic in the teaching of science, and that they were not going quite wild with machines. I have ventured in the course of my remarks on one or two criticisms, but I should like to end on a note of appreciation of the good work which is done by many teachers of physics in our secondary schools.

Sir **Oliver Lodge**, replying on the discussion, said: Well, sir, it is very late, and the audience has diminished since I last spoke, but I think we must all feel that we have heard some very instructive remarks; certainly I feel that. There are, however, just one or two little points on which I may say a word in reply. In the first place some very minor points: There is the term "specific heat," which was often naturally employed by the last speaker. I often feel that we shall never get clear ideas on that subject as long as we use that term at the beginning of things. It is a slang term which we have grown accustomed to. It is an abbreviation for "specific capacity for heat," and the important idea is capacity. One speaker seemed to expect children to know about the refractive index and other properties of the diamond. I think children ought not to be able to answer questions like that. How many of them see diamonds? The students never see them in my laboratory!

The main thing that has been disagreed with in what I said in the opening remarks, is the point which was taken up in the very important observations of experienced teachers like Mr. Bryant and Mr. Ashford—namely, as to their preference for the analytic method. I should like to agree with them, and to a great extent I do agree in practice, but one must discriminate between a systematic course and a general experience. What interests the boys before they enter on a systematic course is the concrete reality, the machine itself. No doubt if this did not exist there would be no interest felt in the principles underlying it. And, therefore, I suppose, the two things ought to go on together, and ought not to be set in opposition to each other; but how intelligibly to dissect out the principles from the complex engine I do not know. I know that the engine is interesting; I used to be fascinated with engines; but I should expect a systematic course to begin with, and build up from principles and not with analysis of the complex thing.

A good deal has been said about experimental mechanics, and it has been stated that this ought to be done by the physicist and not by the mathematician. I do not quite know why; perhaps, only because the physicist will do it better. No doubt he will, because it is his business to do experimental things, but there are so many things in physics more worth his doing than elementary demonstrations either in geometry or in mechanics; and I confess I should like to see the mathematician have to do some experiments, and get in the way of setting the boys to do experiments, even when they are quite elementary ones for juveniles. Let him give them three points, and say, "Now draw a circle through them." Let them invent a method of drawing the circle through three points. That illustration must serve as an illustration for a whole heap of things; and similarly with mechanics, let them be treated not only in the abstract, but in concrete form. At Charterhouse my brother tells me this is now being done; experimental mechanics is now being taught by the mathematical masters, and I welcome it as being so good—for the mathematical masters!

The **President** said: The fact that we finish up this meeting not knowing whether mechanics should be taught by the physicist or by the mathematician, or whether teaching should begin with common objects and proceed to principles or the reverse, proves that there are some points which the discussion has failed to settle, but upon one point there is absolute unanimity, and that is that the course of physics provided in the schools should not be designed to suit only those who ultimately go on to the university, but to suit those who will get in schools the only knowledge of physics they are ever likely to possess.

I ask the members here present to return their thanks to those who have taken part in the discussion, and to Dr. Allen, for having organised it so well. (Applause.)

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